Doctoral Thesis

Loops and Groups

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To the Memory of my Father

Introduction

The dissertation consists of two parts, the first part is devoted to loop theory, the second one presents results in certain areas of classical finite group theory. Though the loop as an algebraic structure differs essentially from groups because the associativity is not required, but using a characterization of multiplication group of loops we can transform loop theoretical problems into group theoretical problems.

The first part is divided into six chapters.

In Chapters 1.2 and 1.3 are results of papers [Cs4], [Cs5].

Chapter 1.4 contains partially our joint work with A. Drápal [CsD1].

In Chapter 1.5 results of paper [Cs7] can be found.

Chapter 1.6 presents results of three joint papers with M. Niemenmaa and K. Myllylä [CsN1], [CsN2] and [CsMN].

The second part contains three chapters.

Chapter 2.1 contains results of two joint papers with M. Asaad [ACs1], [ACs2], partially a joint paper with M. Herzog [CsH], and [Cs1].

Chapter 2.2 presents the results of paper [Cs1], of a joint paper with M. Asaad [ACs3] and another part of paper with M. Herzog [CsH].

Finally Chapter 2.3 consists of the results of [Cs2], of our joint paper with M. Asaad [ACs4] and of a part of the paper with M. Herzog [CsH].

For the better understanding of the introduction concerning the loop theoretical part we give a few definitions in connection with loops.

Q is a loop if it is a quasigroup with neutral element 1.

Let $a \in Q$ be arbitrary. The mappings $L_a : x \to ax$, $R_a : x \to xa$ (for every $x \in Q$) are called left and right translations. Clearly they are permutations on the elements of Q. The permutation group generated by all left and right translations is the multiplication group $\operatorname{Mlt} Q$ of loop Q. The stabiliser of the neutral element in $\operatorname{Mlt} Q$ is the inner mapping group $\operatorname{Inn} Q$ of the loop Q.

In the first part we are working in the multiplication group of loops by using connected transversals, then we tranlate the obtained results into the language of loop theory.

Chapter 1.1 is a short description of roots, historical background, definitions and basic results.

In Chapter 1.2 we study the nilpotency class of loops with abelian inner mapping group. In 1946 Bruck, who laid the foundation of loop theory and defined the multiplication group and inner mapping group, proved in [Br] that the inner mapping group with nilpotency class two is abelian. For a long-standing problem was the converse of Bruck's result:

Problem: Whether every loop with abelian inner mapping group has nilpotency class at most two?

While working on this problem in the early nineties T. Kepka and M. Niemenmaa [NK2], [K1] proved that a finite loop with abelian inner mapping group must be nilpotent. But they did not establish an upper bound on the nilpotency class of the loop, and indeed no such bound is presently known.

For a long time there was no example of nilpotency class greater than two. If the loop is a group, then the converse of Bruck's result holds. We proved with A. Drápal [CsD1] that the LCC (left conjugacy closed) loops (the set of left translations is closed under the conjugation) with abelian inner mapping group are of nilpotency class two. Certain structures of abelian inner mapping group also imply the nilpotency class two [NK2], [CsJK]. Thus for many years the prevailing opinion was that every loop with abelian inner mapping group has nilpotency class at most two.

In 2004 I rejected this conjecture.

Counterexample. Theorem 1.2.13 [Cs5, Statement 4.3]. *I constructed the multiplication group of order* 8192 ($= 2^{13}$) *of a loop* Q *of order* 128 *such that* Q *has nilpotency class three and elementary abelian inner mapping group of order* 2^6*.*

First G. P. Nagy and P. Vojtěchovský [NV1] by using GAP analysed the loop structure of my example and they could construct by greedy algorithm another loop of order 128 with nilpotency class 3.

Recently A. Drápal and P. Vojtěchovský [DV] using GAP package LOOPS constructed the multiplication table of this loop Q. By analysing the structure of the counterexample Q, they developed a method by which they were able to construct a class of other counterexamples, among which the *very natural* (as they expressed it in their paper) was my counterexample.

Originally I tried to prove the conjecture. By introducing the notion of the so called *nice subclass* I analysed the structure of the multiplication group of counterexample of minimal order, see Proposition 1.2.11 [Cs4, Proposition 3.4]. Finally using the properties of this nice subclass I could construct the counterexample. I even obtained further sufficient conditions for nilpotency class two with particular attention to the structure of the normal closure of the inner mapping group, see Proposition 1.2.11 [Cs4, Proposition 3.4], Theorem 1.2.18 [Cs4, Theorem 3.5], Corollary 1.2.22 [Cs4, Corollary 4.3], Theorem 1.2.19 [Cs4, Theorem 3.6], Corollary 1.2.23 [Cs4, Corollary 4.4], Theorem 1.2.15 [Cs4, Theorem 3.7], Corollary 1.2.20 [Cs4, Corollary 4.5] Theorem 1.2.17 [Cs4, Theorem 3.1], Corollary 1.2.21 [Cs4, Corollary 4.6].

While studying the minimal counterexample and the properties of *nice subclass* we have got some partial answer to another question closely related to the nilpotency class:

Question: Which abelian groups can (or cannot) occur as inner mapping groups of a loop?

The question which finite abelian groups are possible as inner automorphism groups of groups was completely solved by Baer [Bae4]. The result is as follows:

Let G be a finite abelian group and let $G = C_1 \times C_2 \times \cdots \times C_n$ be the direct product of cyclic groups such that $|C_{i+1}|$ divides $|C_i|$ $(i = 1, \ldots, n-1)$. Then there exists a group H such that $\operatorname{Inn} H \cong G$ if and only if $n \geq 2$ and $|C_1| = |C_2|$.

The obtained results show that the situation in loop theory is similar as concerns the structure of finite abelian inner mapping group.

Even the conjecture – as M. Niemenmaa formulated it in [N1] – is the following:

Conjecture: If Q is a loop and $\operatorname{Inn} Q = C_1 \times C_2 \times \cdots \times C_n$ is a direct product of cyclic subgroups such that $|C_{i+1}|$ divides $|C_i|$, for every $1 \leq i \leq n-1$, then $n \geq 2$ and $|C_1| = |C_2|$.

The direction of research is mainly determined by the above mentioned Baer theorem.

The topic of this Chapter 1.3 is this problem, which was originally motivated by T.Kepka and M. Niemanmaa's result [NK1], they proved the non-existence of nonassociative loop with nontrivial cyclic inner mapping group. Then several negative answers appeared to this question: [N3], [N4], [CsJK], [K1], [CsK].

I gave some generalizations of these earlier results, see Theorem 1.3.1 [Cs4, Theorem 3.10], Theorem 1.3.2 [Cs4, Theorem 3.11], Corollary 1.3.3 [Cs4, Corollary 4.1]. Recently M. Niemenmaa [N8] extended my statements, his proof depends heavily on my result.

Chapter 1.4 is also related to the nilpotency class. In case of left conjugacy closed, LCC loops with abelian inner mapping group we showed the nilpotency class two in [CsD1]. Here I present the original group theoretical proof (see Theorem 1.4.3) in the multiplication group using the technique of H-connected transversals.

In Chapter 1.5 we study the properties of loops that are abelian groups over the nucleus. This study is motivated by our earlier results. In case of Buchsteiner loops we proved [CsDK], [CsD3] that the factorloop over the nucleus Q/N is an abelian group and the factorloop over the center $Q/Z(Q)$ is conjugacy closed. In case of an arbitrary loop Q we got some sufficient conditions for that $Q/Z(Q)$ is a conjugacy closed loop provided Q/N is an abelian group [CsD3, Theorem 3.1, Proposition 3.2]. The continuation of this study resulted in this chapter. I improved and generalized these results, see Proposition 1.5.9 [Cs7, Proposition 3.7], Proposition 1.5.10 [Cs7, Proposition 3.8], Proposition 1.5.11 [Cs7, Proposition 3.9], Proposition 1.5.13 [Cs7, Proposition 3.11], Proposition 1.5.14 [Cs7, Proposition 3.12].

In case of abelian inner mapping group with the property Q/N is an abelian group I obtained an upper bound three for nilpotency class, see Theorem 1.5.16 [Cs7, Theorem 3.14]. Then I have found some sufficient conditions for nilpotency class three of multiplication group provided Q/N is an abelian group, see Theorem 1.5.18 [Cs7, Theorem 3.16]. Finally, using these results I gave a structural description of Buchsteiner loops with abelian inner mapping group, see Theorem 1.5.22 [Cs7, Theorem 3.20].

The topic of the last Chapter 1.6 of the first part is the solvability of loops. It is a known result that in case of finite loops from the solvability of multiplication group follows the solvability of the loop [Ve1].

Question: which properties of inner mapping group imply the solvability of the multiplication group?

In the nineties Kepka and Niemenmaa proved the solvability of a multiplication group in case of abelian inner mapping group [NK3]. M. Niemenmaa could prove [N5] that $\mathrm{Mlt}\, Q$ is solvable if $|\mathrm{Inn}\, Q| = 6$, later in case $\mathrm{Inn}\, Q$ is a dihedral 2-group [N7]. The more general problem was if the order of $\mathrm{Inn}\, Q$ is the product of two different primes p and q. First this problem was solved for very special primes by using the classification of finite simple groups [N6], [MN].

With M. Niemenmaa we obtained the proof of solvability for $|\mathrm{Inn}\, Q| = 2p$, where p is an arbitrary odd prime, see Theorem 1.6.9 [CsN1, Theorem 2.4] and Theorem 1.6.10 [CsN1, Theorem 3.1], later for $|\mathrm{Inn}\, Q| = pq$ with $p > q > 2$, $p = 2q^m + 1$, see Theorem 1.6.31 [CsN2, Theorem 3.1], and in case if $\mathrm{Inn}\, Q$ is a dihedral group of order $2p^n$, see Theorem 1.6.19 [CsMN, Theorem 3.6] and Theorem 1.6.20 [CsMN, Theorem 4.2].

The second part, the *group part* consists of three chapters.

Chapter 2.1 analyses the influence of minimal subgroups on the structure of finite groups.

The development of this area was motivated by Buckley's result [Bu], namely if every minimal subgroup of a group of odd order is normal, then the group is supersolvable. After Buckley's result Kegel [Ke] introduced the notion of S-quasinormality (a subgroup is S-quasinormal, if it permutes with every Sylow subgroup of the whole group). Several authors studied the influence of S-quasinormality of some subgroups which ensures the supersolvability of the group [ARS], [Sha], [Sr]. Using formation theorem the results have been extended for saturated formations containing the class of supersolvable groups [Yo1], [Yo2], [La], [AsBP].

In our studies with M. Asaad we supposed the S-quasinormality of subgroups of minimal order or of order 4 of the Fitting subgroup of some solvable normal subgroup, so we received a necessary and sufficient condition for supersolvability, see Theorem 2.1.1 [ACs1, Theorem].

Later Li and Wang generalized our result [LW1], omitting the solvability of the normal subgroup, and instead of the Fitting subgroup, they supposed similar conditions for the generalized Fitting subgroup.

In [Cs1] I presented a characterization of a solvable group G under the assumption that every subgroup of $F(G)$ of prime order or order 4 is S-quasinormal in G, see Theorem 2.1.9 [Cs1, Theorem 5] and Theorem 2.1.10 [Cs1, Theorem 4]. Later with M. Asaad we studied the situation when the quaternion group of order 8 is not involved in G, but we did not suppose the solvability of G, see Theorem 2.1.11 [ACs2, Theorem 1.1].

Many new generalizations of our results were obtained by introducing the notion of S-quasinormally embedded subgroup [LW2], and a weaker new embedding property, namely the 3-permutability (or Σ-permutability) [AsH], [HLL], [WW].

The second section of this chapter is devoted to the influence of $\mathcal{H}$-property of some subgroups on the structure of finite groups. In 2000 Herzog, Bianchi et al. [BMHV] introduced the notion of $\mathcal{H}$-subgroup. A subgroup K of a group G is called an $\mathcal{H}$-subgroup of G if the following condition is satisfied:

$$N_G(K) \cap K^g \subseteq K \quad \text{for every} \quad g \in G.$$

With M. Herzog we first characterized those groups for which every cyclic subgroup of prime order or order 4 possesses the $\mathcal{H}$-property, see Theorem 2.1.20 [CsH, Theorem 10]. Then we gave sufficient conditions for supersolvability by requiring the $\mathcal{H}$-property of certain minimal subgroups. See Theorem 2.1.21 [CsH, Theorem 11], Theorem 2.1.22 [CsH, Theorem 12].

Later M. Asaad introducing the notion of weakly supersolvable p-groups [As2] and Li Yangming introducing the notion of NE-subgroups [Ya] received new sufficient conditions for supersolvability. Their results heavily depend on our statements on $\mathcal{H}$-subgroups.

In Chapter 2.2 our subject is the general study of supersolvability. In previous chapter we presented some sufficient conditions for supersolvability that describe a relatively small subclass of supersolvable groups.

Here we give a natural factorization of supersolvable groups, see Theorem 2.2.1 [Cs1, Theorem 1]. As a corollary of this result we get another characterization of supersolvable groups based on the structure of Fitting subgroup, see Theorem 2.2.4 [Cs1, Theorem 2], Theorem 2.2.5 [Cs1, Theorem 3].

Recently with M. Asaad we improved and extended this natural factorization, relying on Σ-permutability and on the generalized Fitting subgroup. See Theorem 2.2.6 [ACs3, Theorem 1.1], Theorem 2.2.7 [ACs3, Theorem 1.2] and Theorem 2.2.8 [ACs3, Theorem 1.3].

Using the original factorization theorem with M. Herzog we gave a characterization of supersolvable SSA-groups (all Sylow subgroups are abelian). See Theorem 2.2.17 [CSH, Theorem 19].

Later M. Asaad generalized this result [As1].

The subject of the last Chapter 2.3 of the second part is the study of solvable T-groups and T^*-groups.

A group G is called a T-group, if its every subnormal subgroup is normal in G.

We have to mention results of Best and Taussky [BT], later the characterization of solvable T-groups [Za], then Gaschütz's theorem on solvable T-groups [Ga].

With M. Herzog we gave a structural description of solvable T-groups, based on the assumption that certain subgroups possess the $\mathcal{H}$-property, see Theorem 2.3.6 [CsH, Theorem 14] and Corollary 2.3.7 [CsH, Corollary 15], Corollary 2.3.8 [CsH, Corollary 16].

A subgroup is said to be permutable if it permutes with every subgroup of the group. A group is called a PT-group if the permutability is transitive, which means by Ore [Or] that every subnormal subgroup is permutable. The definitive result on solvable PT-groups is due to Zacher.

The concept of T^*-groups (or PST-groups) was introduced by Kegel [Ke]. I remind the reader that a subgroup of a group G is π-quasinormal in G (or S-quasinormal) if it permutes with every Sylow subgroup of G. A group is called T^*-group (or PST-group), if the π-quasinormality (or S-quasinormality) is transitive, which means by Kegel that every subnormal subgroup of a group G is π-quasinormal (or S-quasinormal) in G.

The structure of solvable PST-groups was determined by Agrawal [Ag]. With M. Asaad we extended some results concerning T-groups to T^*-groups and we gave some structural description of solvable T^*-groups, one of them by using the pronormality of certain subgroups, see Theorem 2.3.14 [ACs4, Theorem 1], Theorem 2.3.15 [ACs4, Theorem 2], Theorem 2.3.16 [ACs4, Theorem 3], Theorem 2.3.19 [ACs4, Theorem 6], Theorem 2.3.20 [ACs4, Theorem 7], Theorem 2.3.21 [ACs4, Theorem 8], Theorem 2.3.22 [ACs4, Theorem 9].

Later I generalized Zacher's theorems concerning solvable T-groups for solvable T^*-groups, see Theorem 2.3.23 [Cs3, Theorem 1]. Then by analysing the properties of Sylow subgroups I obtained another characterization of this subclass of finite groups. See Theorem 2.3.26 [Cs3, Theorem 3], Theorem 2.3.27 [Cs3, Theorem 4] and Theorem 2.3.25 [Cs3, Theorem 2].

Six years later this latter result was obtained by Ballester-Bolinches and Esteban-Romero in another form by introducing the notion of Y_p-group [BR].

By studying new properties of PT and PST-groups, this topic is the subject of extensive research. See [ABRP], [ABP], [Ra2], [BRR], [MS], [BRP].

LOOPS

A quasigroup $(Q, \cdot)$ is a set Q together with a binary operation $\cdot$ such that for each $a, b \in Q$, the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for every $x, y \in Q$. A quasigroup Q is a loop if it has a so-called neutral element $1 \in Q$ satisfying $1x = x \cdot 1 = x$ for every $x \in Q$. Consequently the loops can be considered as nonassociative versions of loops.

We denote the solutions x and y of equations $ax = b$, $ya = b$ by $x = a \setminus b$ and $y = b \,/\, a$.

The theory of loops is a fairly young discipline which has its roots in finite projective geometries and Latin squares.

Finite loops (like finite quasigroups) can be expressed by their multiplication tables.

The smallest loops are groups, and the smallest loop that is not a group is of order 5. There are, up to isomorphism, five such loops, one of them is the following:

1	2	3	4	5
2	1	4	5	3
3	5	1	2	4
4	3	5	1	2
5	4	2	3	1

It is clear that multiplication tables of quasigroups are *Latin squares* (tables with n columns and n rows, where each row and column permutes a given set of n elements – usually $1, \ldots, n$) and that multiplication tables of loops correspond to *normalized Latin squares* (Latin squares where the first row and the first column contain $1, 2, \ldots, n$ in their naturally increasing order).

In his famous work *Grundlagen der Geometrie* Hilbert considered a projective plane as a system of axioms for the incidence relations between points and lines. This approach connects certain concepts of algebra with those in geometry and leads to the study of such algebraic structures as alternative division rings and loops.

The first step towards general theory of loops and quasigroups can be found in books of Ernst Schroeder (1873 and 1890). The first attempts to establish a systematic theory can be attributed to K. Suškevič. The first two major works [M1, M2] were introduced by Ruth Moufang in 1933 and 1934. She and Geritt Bol studied loops with additional properties inspired by certain geometrical configurations, namely they studied the translations in hyperbolic plane and nonzero real octonions. Later on (1939–1944) A. Albert [Al1, Al2] and R. Baer [Bae1] considered algebraic structures called quasigroups. Finally in 1946 R. Bruck [Br] laid the foundation of loop theory. In this article Bruck defined the concepts of the multiplication group and the inner mapping group of a loop, thus creating a link between loop theory and group theory. In the 1960's, 1970's and 1980's several special cases concerning loops were investigated by Baer, Glauberman, Doro, Smith and Liebeck. Recently the subject has been applied in quite new areas (e.g. J. H. Conway used a special loop in the construction of the Fischer–Griess Monster in finite group theory) and it became a thriving branch of mathematics.

1.1. Definitions, notions, basic results

The following definitions and results are mainly based on Bruck's important paper [Br] *Contributions to the theory of loops.*

Definition. If the operation on loop Q is commutative, then Q is a *commutative loop.*

Definition. A subset H of a loop Q is called *subloop* of Q if it is also a loop with respect to the same operation. We denote this by $H \leq Q$.

Definition. A subloop H of a loop Q is called *normal subloop* of Q if

$$aH = Ha, \quad a(bH) = (ab)H, \quad (Hb)a = H(ba)$$

for every $a, b \in H$. We denote it by $H \trianglelefteq Q$.

Definition. If H is a normal subloop of the loop Q, the separate cosets Hx form a loop with operation $(Hx)(Hy) = H(xy)$. This latter loop is called the *factorloop* of Q over H and we denote it by Q/H.

Definition. The *left*, *middle* and *right nucleus* of a loop Q are defined, respectively by

$$\begin{aligned}
N_\lambda = N_\lambda(Q) &:= \{a \in Q \mid (ax)y = a(xy) \text{ for all } x, y \in Q\},\\
N_\mu = N_\mu(Q) &:= \{a \in Q \mid x(ay) = (xa)y \text{ for all } x, y \in Q\},\\
N_\varrho = N_\varrho(Q) &:= \{a \in Q \mid x(ya) = (xy)a \text{ for all } x, y \in Q\}.
\end{aligned}$$

N_λ, N_μ, N_ϱ are subloops of Q, but generally they are not normal subloops of Q. The intersection

$$N = N(Q) = N_\lambda \cap N_\mu \cap N_\varrho$$

is called the *nucleus* of Q, and it is a subloop of Q.

The *centre* of Q:

$$Z(Q) = \{a \in N \mid xa = ax \text{ for all } x \in Q\}.$$

In other words the centre $Z(Q)$ includes all such elements of Q that are commutative and associative with all elements of Q. It can be proved that $Z(Q)$ is a normal subloop of Q.

The *commutator* for $x, y \in Q$ is $[x, y] = (yx) \setminus (xy)$.

The *associator* for $x, y, z \in Q$ is $[x, y, z] = (x(yz)) \setminus ((xy)z)$.

The *associator subloop* $A(Q)$ of Q is the least normal subloop of Q such that $Q/A(Q)$ is a group.

The *commutator-associator subloop* Q' of loop Q is the least normal subloop of Q such that Q/Q' is an abelian group.

Bruck in [Br] defined the solvability and nilpotency of loops as follows:

Definition. A loop Q is *solvable* if it has a series

$$1 = Q_0 \leq Q_1 \leq \cdots \leq Q_n = Q,$$

where $Q_{i-1} \trianglelefteq Q_i$ and the factorloop Q_i/Q_{i-1} is an abelian group for every $1 \leq i \leq n$.

Definition. Let Q be a loop. If we set $Z_0 = 1$, $Z_1 = Z(Q)$ and factorloop $Z_i/Z_{i-1} = Z(Q/Z_{i-1})$, then we obtain a series of normal subloops of the loop Q. If Z_{n-1} is a proper subloop of Q, but $Z_n = Q$, then we say that loop Q is *centrally nilpotent* of class n, i.e. $\operatorname{cl} Q = n$.

Also Bruck in [Br] defined the multiplication group of a loop and he was the first to investigate the structure of loops by using group theory.

Definition. Let Q be a loop, $a \in Q$ is arbitrary. The mappings

$$L_a : x \to ax \qquad R_a : x \to ax \quad \text{ for every } \ x \in Q$$

are called *left* and *right translation* of a respectively.

Clearly L_a and R_a are permutations on the elements of Q.

The permutation group generated by left and right translations is the *multiplication group* of the loop Q. Denote it by $\operatorname{Mlt} Q$:

$$\operatorname{Mlt} Q = \langle L_a,\ R_a \mid a \in Q \rangle.$$

The stabilizer of the neutral element 1 in $\operatorname{Mlt} Q$ is the *inner mapping group* or *inner permutation group* of the Q and we denote it by $\operatorname{Inn} Q$.

$$\operatorname{Inn} Q = \operatorname{Stab}(1).$$

It is easy to see that a loop Q is a group if and only if for each $a, b \in Q$ there exists $c \in Q$ such that $L_a L_b = L_c$.

Remark that in case when Q is a group $\operatorname{Inn} Q$ is the usual inner automorphism group of Q.

$|Q|$ is the order of the loop Q i.e. its cardinal number.

Clearly if Q is a finite loop, then

$$|Q| = |\operatorname{Mlt} Q : \operatorname{Inn} Q|.$$

The properties of multiplication groups

Suppose Q is a loop. Denote

$$A = \{L_a \mid a \in Q\}, \quad B = \{R_a \mid a \in Q\}.$$

It can be shown:

1) A and B are left transversals to $\operatorname{Inn} Q$ in $\operatorname{Mlt} Q$.
2) The commutator subgroup $[A, B] \leq \operatorname{Inn} Q$.
3) $\langle A, B\rangle = \operatorname{Mlt} Q$.
4) $\mathrm{core}_{\operatorname{Mlt} Q} \operatorname{Inn} Q = 1$.

(${\mathrm{core}_{\operatorname{Mlt} Q}} \operatorname{Inn} Q = 1$ means the largest normal subgroup of $\operatorname{Mlt} Q$ in $\operatorname{Inn} Q$.)

The corresponding situation in groups:

Let G be a group, H is a subgroup of G. Suppose there exist A and B left transversals to H in G. We say A and B are H-connected transversals, if the commutator subgroup $[A, B] \leq H$. If $A = B$ and $[A, A] \leq H$, then A is called H-selfconnected transversal. In fact H-connected transversals are both left and right transversals. Denote the core of H in G by $L_G(H)$, i.e. the largest normal subgroup of G contained in H.

Given this, a question arises as to which kinds of groups can be multiplication groups of loops.

The following theorem which was proved by Niemenmaa and Kepka [NK1, Theorem 4.1] describes the relationship between multiplication group of loops and connected transversals:

Theorem 1.1.1. *A group G is isomorphic to the multiplication group of a loop Q if and only if there exist a subgroup H satisfying $L_G(H) = /\mathrm{core}_G H/ = 1$ and H-connected transversals A and B such that $G = \langle A, B\rangle$.*

We remark that if Q is a loop then we can choose $G = \operatorname{Mlt} Q$, $H = \operatorname{Inn} Q$ and A and B the set of left and right translations respectively.

The construction of the loop from its multiplication group:

If G has a subgroup H and H-connected transversals A and B satisfying the conditions of the theorem, then we can construct a loop K whose elements are the left cosets of H in G. For $a, b \in A$ the operation $(aH)(bH) = cH$ if and only if $abH = cH$ with $c \in A$.

In case $A = B$, the corollary of this theorem [NK1, Corollary 4.1]:

Corollary 1.1.2. *A group G is isomorphic to the multiplication group of a commutative loop if and only if there exist a subgroup H of G satisfying $L_G(H) = 1$ and H-selfconnected transversal A such that $G = \langle A\rangle$.*

Using this characterization theorem our loop theoretical problems can be transformed into purely group theoretical problems. Many properties of loops can be reduced to the properties of connected transversals in the multiplication group.

Definition. The *left multiplication group* and *right multiplication group* of the loop Q are defined respectively:

$$\mathcal{L} = \langle L_a \mid a \in Q \rangle,$$
$$\mathcal{R} = \langle R_a \mid a \in Q \rangle.$$

Denote

$$\mathcal{L}_1 = \mathcal{L} \cap \operatorname{Inn} Q,$$
$$\mathcal{R}_1 = \mathcal{R} \cap \operatorname{Inn} Q.$$

They are called *left* and *right* inner mapping group of the loop respectively.

Denote

$$L(x,y) = {L_{xy}}^{-1} L_x L_y, \qquad T_x = R_x^{-1} L_x,$$
$$R(x,y) = {R_{yx}}^{-1} R_x R_y \qquad \text{for every } \ x, y \in Q.$$

We have

Proposition 1.1.3.

$$\mathcal{L}_1 = \langle L(x,y) \mid x, y \in Q \rangle,$$
$$\mathcal{R}_1 = \langle R(x,y) \mid x, y \in Q \rangle,$$
$$\operatorname{Inn} Q = \langle \mathcal{L}_1, \mathcal{R}_1, T_x \mid x \in Q \rangle.$$

We have that the centralizer of the left and right multiplication group in the whole multiplication group is the following:

Proposition 1.1.4.

$$C_{\operatorname{Mlt} Q}(\mathcal{L}) = \{R_a \mid a \in N_\varrho\},$$
$$C_{\operatorname{Mlt} Q}(\mathcal{R}) = \{L_a \mid a \in N_\lambda\}.$$

Definition. $C(Q) = \{x \in Q \mid L_x = R_x\}$.

Proposition 1.1.5. $Z(Q) = N \cap C(Q)$.

We need another description of normal subloop with the aid of multiplication group of the loop Q:

A well-known result:

Proposition 1.1.6. *A subloop S of a loop Q is normal in Q if and only if all the elements of inner mapping group $\operatorname{Inn} Q$ are permutations on the subloop S.*

Using $\mathrm{Mlt}\, Q = A \cdot \mathrm{Inn}\, Q$ it follows immediately that the normal subloops of Q are exactly the blocks of $\mathrm{Mlt}\, Q$ which contain the neutral element. Hence the normal subloops of Q correspond to those subgroups of $\mathrm{Mlt}\, Q$ that contain $\mathrm{Inn}\, Q$.

It can be verified easily that each of these subgroups are of form $M(S)\mathrm{Inn}\, Q$ where $M(S) = \langle L_s, R_s \mid s \in S \rangle$.

Definition. The *multiplication group* $\mathrm{Mlt}\,(Q/S)$ *of the factorloop* Q/S is the image of $\mathrm{Mlt}\, Q$ on its action on the cosets of S. The kernel of this action is

$$K = \{\varphi \in \mathrm{Mlt}\, Q \mid \varphi(xS) = xS \text{ for every } x \in Q\}.$$

We have $K = \mathrm{core}_{\mathrm{Mlt}\, Q}(M(S)\mathrm{Inn}\, Q)$.

In the language of connected transversals the multiplication group of a factorloop can be obtained in the following way.

Proposition 1.1.7. *Let Q be a loop and S is a normal subloop of Q. Denote $G = \mathrm{Mlt}\, Q$, $H = \mathrm{Inn}\, Q$, $A = \{L_a \mid a \in Q\}$, $B = \{R_a \mid a \in Q\}$. Let $C = \langle L_s, R_s \mid s \in S \rangle$ and $L = \mathrm{core}_G CH$. Denote f the natural homomorphism of G onto G/L. Then $\mathrm{Mlt}\,(Q/S) \cong G/L$, $\mathrm{Inn}\,(Q/S) \cong LH/L$, furthermore $f(A)$ and $f(B)$ are $f(H)$-connected transversals.*

1.2. Abelian inner mappings and nilpotency class

As it is well known, a group is of nilpotency class at most two if and only if its inner automorphism group is abelian. In 1946 Bruck published a long paper [Br] that influenced the development of loop theory for decades, in which he proved that a loop of nilpotency class two possesses an abelian inner mapping group.

The converse problem of Bruck's result: Is every finite loop (even infinite) with abelian inner mapping group nilpotent of class at most two?

While working on this problem, Kepka and Niemenmaa [NK2, Cor. 6.4] proved that a finite loop with abelian inner mapping group must be nilpotent. (Kepka later improved upon this result and showed that if the inner mapping group is abelian and finite, then the loop is nilpotent [K1].) But they did not establish an upper bound on the nilpotency class of the loop, and, indeed, no such bound is presently known.

For a long time there was no example of a nilpotency class greater than two. In fact, it seems that for many years the prevailing opinion has been that all such loops have to be of nilpotency class two. This seems to have been well substantiated since if the loop is a group, we clearly get this restriction on the nilpotency class. Some well-behaved classes of loops fulfil this restriction, too. It was proved recently by Drápal and Csörgő [CsD1, Theorem 2.7] that left conjugacy closed loops with the abelian inner mapping groups are of nilpotency class two. The special structure of abelian inner mapping group also implies the nilpotency class two of the loop (see in the next chapter).

Thus many experts believed that the converse of Bruck's result holds. But in 2004 (the result was published in 2007) using the technique of H-connected transversals I was able to construct a counterexample to this long-standing conjecture. I constructed – by hand – the muliplication group (of order 8192) of a loop of order 128 of nilpotency class three with abelian inner mapping group.

G. P. Nagy and P. Vojtěchovský [NV1] by using GAP analysed the loop structure of my counterexample and they could construct by greedy algorithm another loop of order 128 with nilpotency class 3.

Recently A. Drápal and P. Vojtěchovský [DV] using computer GAP package LOOPS constructed the multiplication table of my counterexample loop. By analyzing the structure of the counterexample loop Q they developed a method by which they were able to construct a class of other examples among which the simplest one was my original loop Q, even they state this Q is very natural among loops of nilpotency class three with abelian inner mapping group.

Fresh result: G. P. Nagy and P. Vojtěchovský constructed a Moufang 2-loop (of order 2^{14}) [NV2] of nilpotency class three with abelian inner mapping group. At the same time they showed: Moufang loops of odd order with abelian inner mapping groups have nilpotency class at most two.

A short time ago A. Drápal and M. Kinyon [DK] constructed a Buchsteiner loop of order 128 of nilpotency class three with abelian inner mapping group.

My original goal was to prove the converse of Bruck's result and I was gradually accumulating the properties of the multiplication group of a minimal counterexample so that its existence could be refuted, but I ended up constructing a counterexample.

I was working in the multiplication group using the technique of H-connected transversals and introducing the notion of the nice subclass. During the study I obtained some sufficient conditions for nilpotency class two which are generalizations of our earlier results.

Converse problem of Bruck's result

Problem 1.2.1. Let Q be a loop with abelian inner mapping group. Does it follow that the nilpotency class of Q is at most two?

Preliminary results

Theorem 1.2.2. *A loop Q is centrally nilpotent of class at most $n \geq 1$ if and only if the inner mapping group $\operatorname{Inn} Q$ is subnormal of depth at most n in the multiplication group $\operatorname{Mlt} Q$.*

Proof. See [K1, Proposition 4.1] and [KPh]. □

Corollary 1.2.3. *A loop Q is centrally nilpotent of class at most two if and only if $(\operatorname{Mlt} Q)' \leq N_{\operatorname{Mlt} Q}(\operatorname{Inn} Q)$.*

Proof. By Theorem 1.2.2 Q is centrally nilpotent of class at most two if and only if $N_{\operatorname{Mlt} Q}(\operatorname{Inn} Q) \trianglelefteq \operatorname{Mlt} Q$. Using $[A, B] \leq \operatorname{Inn} Q$ this latter normality means that the factor group $\operatorname{Mlt} Q / N_{\operatorname{Mlt} Q}(\operatorname{Inn} Q)$ is abelian and this implies our statement. □

By using Theorem 1.1.1 and Corollary 1.2.3 we can transform our loop-theoretical problem – converse of Bruck's result – to a group-theoretical one in the following way:

Problem 1.2.1*. Assume G is a finite group with the following properties: there is an abelian subgroup H of G, there exist A and B H-connected left transversals to H in G such that $\langle A, B\rangle = G$, furthermore $\operatorname{core}_G H = 1$. Do these conditions imply $G' \leq N_G(H)$?

In some cases we study this problem without supposing $\operatorname{core}_G H = 1$. Denote $L_G(H) = \operatorname{core}_G H$.

Thus in this chapter:

> *G is a finite group with abelian proper subgroup H. There exist A and B H-connected left transversals, i.e. A and B are left transversals to H with $[A, B] \leq H$, furthermore $\langle A, B\rangle = G$.*

We need the following:

Lemma 1.2.4. *If $L_G(H) = 1$, then $N_G(H) = H \times Z(G)$ and $Z(G) \subseteq A \cap B$.*

For the proof see [NK1, Proposition 2.7], [KN1, Lemma 1.4]. □

Lemma 1.2.5 (see [NK2, Proposition 6.3])**.** *If G is a finite group such that $G = \langle A, B\rangle$ and H is abelian, then H is subnormal in G.*

Theorem 1.2.6 (see [NK2, Theorem 4.1])**.** *If H is abelian, then G is solvable.*

Lemma 1.2.7 (see [NK1, Lemma 2.8])**.** *Let $H \leq G$ and let A and B H-connected transversals in G. Consider a normal subgroup N of G, put $L = L_G(HN)$, and denote by f the natural homomorphism of G onto G/L. Then $f(A)$ and $f(B)$ are $f(H)$-connected transversals in G/L.*

Properties of the multiplication group of minimal counterexample

Main results

Proposition 1.2.8 [Cs4, Proposition 3.1]**.**

i) *Let G_1 be a subgroup of G which contains H. If $H < N \trianglelefteq G_1$, then G_1/N is abelian.*

ii) $G_0 = G'H$.

iii) *If $G' \not\leq N_G(H)$, then $G_0 \neq G$.*

Proof. i) Let aN and bN be arbitrary elements of G_1/N with $a \in A$, $b \in B$ (A and B are left transversals to H). Then using $[A, B] \leq H$ we get $[aN, bN] \leq H < N$, consequently G_1/N is abelian.

ii) Clearly $G'H \geq G_0$. By i) G/G_0 is abelian, which gives $G_0 \geq G'$.

iii) Clearly $N_G(H) \neq G$. The subnormality of H in G (see Lemma 1.2.5) implies that there exists a normal subgroup W of G such that $H < W \neq G$. By i) G/W is abelian, whence $W \geq G'$. By ii) we get $W \geq G_0$. Since $W \neq G$ it follows $G_0 \neq G$.

□

Proposition 1.2.9 [Cs4, Proposition 3.2]**.** *Suppose $L_G(H) = 1$. Then the following statements hold:*

i) $A \cap H = B \cap H = \{e\}$.

ii) $Z(G) \neq 1$.

iii) $Z(G_0) = (Z(G) \cap G_0) \times (Z(G_0) \cap H),\ \ Z(G) \cap G_0 \neq 1,\ \ Z(G_0) \neq 1$.

iv) $\mathrm{core}_G((Z(G) \cap G_0)H) = Z(G_0)$.

v) $(Z(G) \cap G_0)A = A, \quad (Z(G) \cap G_0)B = B$.

vi) *If $[A, B] \leq Z(G_0) \cap H$, then $AZ(G_0) \trianglelefteq G$ and $BZ(G_0) \trianglelefteq G$.*

Proof. i) Suppose $a \in A \cap H$ and $a \neq e$. Then using $[A, B] \leq H$ we get $a^{-1}b^{-1}ab \in H$, whence $a^b \in H$ for every $b \in B$. Since $G = BH$ then $L_G(H) \neq 1$, a contradiction. In a similar way $B \cap H = \{e\}$.

ii) By Lemma 1.2.4 $N_G(H) = H \times Z(G)$. The subnormality of H (see Lemma 1.2.5) gives our statement.

iii) The subnormality of H in G (see Lemma 1.2.5) implies the subnormality of H in G_0, too. Hence $N_{G_0}(H) \neq H$. Since $N_G(H) = Z(G) \times H$ by Lemma 1.2.4, we get $N_{G_0}(H) = (Z(G) \cap G_0) \times H$, consequently $Z(G) \cap G_0 \neq 1$. By using $Z(G_0) \leq C_G(H) = Z(G) \times H$ we can conclude $Z(G_0) = (Z(G) \cap G_0)(Z(G_0) \cap H)$. $Z(G) \cap G_0 \neq 1$ implies $Z(G_0) \neq 1$.

iv) Denote $U_1 = \mathrm{core}_G((Z(G) \cap G_0)H)$ and $H_1 = U_1 \cap H$. Then $U_1 = (Z(G) \cap G_0) \times H_1$. As H is abelian $H \leq C_G(U_1)$ holds. We have $U_1 \trianglelefteq G$, whence $H^g \leq$

$C_G(U_1)$ follows for every $g \in G$. The definition of G gives $U_1 \leq Z(G_0)$. iii) implies $U_1 \geq Z(G_0)$ and we get $U_1 = Z(G_0)$.

v) Let $z \in Z(G) \cap G_0$ and $a \in H$. Then $za \in a_0 H$ for some $a_0 \in A$. Using $[A, B] \leq H$ it follows $(a_0^{-1} za)^b \in H$ for every $b \in B$. Since $BH = G$ we get $a_0^{-1} za \in L_G(H)$, hence $a_0 = za$. Thus $(Z(G) \cap G_0)A = A$, in a similar way $(Z(G) \cap G_0)B = B$.

vi) Let $\alpha_1 z_1$, $\alpha_2 z_2$ be arbitrary elements of AZ_0 with $\alpha_1, \alpha_2 \in A$, $z_1, z_2 \in Z(G_0)$. By using $Z(G_0) \trianglelefteq G$ we get $\alpha_1 z_1 \alpha_2 z_2 = \alpha_1 \alpha_2 z_3$ where $z_3 \in Z(G_0)$. Let $\alpha \in A \cap \alpha_1 \alpha_2 H$. Let $\beta \in B$ be arbitrary. Then using $[A, B] \leq Z(G_0) \cap H$ and $Z(G_0) \trianglelefteq G$ we get $(\alpha^{-1}\alpha_1\alpha_2)^\beta \in \alpha^{-1}\alpha_1\alpha_2 Z(G_0)$. As $\alpha^{-1}\alpha_1\alpha_2 = h^* \in H$ we have $(h^*)^\beta \in h^* Z(G_0)$ for arbitrary $\beta \in B$. Using $G = BH$ we can conclude $h^* \in \mathrm{core}_G(HZ(G_0))$. Proposition 1.2.9 iii) and iv) imply $h^* \in Z(G_0)$. Thus the product of any two elements of AZ_0 is in AZ_0. Since G is finite then $AZ(G_0) \leq G$. From $[A, B] \leq Z(G_0) \cap H$ and $Z(G_0) \trianglelefteq G$ we get $B \subseteq N_G(AZ(G_0))$. $G = \langle A, B \rangle$ implies $AZ(G_0) \trianglelefteq G$.

Similarly we get $BZ(G_0) \trianglelefteq G$. $\square$

Proposition 1.2.10 [Cs4, Proposition 3.3]**.** *Let $Z^* \leq Z(G) \cap G_0$. Suppose $Z^* H \trianglelefteq G_0$ and $L_G(H) = 1$. Denote $U = \mathrm{core}_G(Z^* H)$ and $H^* = U \cap H$. Then the following statements hold:*

i) $U = Z^* \times H^*$, $\ H^* \leq Z(G_0)$.

ii) *If $a \in A \cap G_0$ and $b \in B \cap aH$, then $a^{-1}b \in H^*$.*

iii) $[A \cap G_0, B] \leq H^*$, $\ [B \cap G_0, A] \leq H^*$.

iv) *$A \cap G_0$ and $B \cap G_0$ are abelian subgroups of G.*

v) *$(A \cap G_0)H^*$ is abelian normal subgroup of G, $Z^* H^* \leq (A \cap G_0)H^*$ and $(A \cap G_0)H^* = (B \cap G_0)H^*$.*

vi) $G_0' \leq H^* Z^* \leq (A \cap G_0)H^*$.

vii) $\mathrm{core}_{G_0} H = Z(G_0) \cap H$.

Proof. i) Clearly $U = Z^* \times H^*$. As H is abelian $H \leq C_G(U)$ holds. Since $U \trianglelefteq G$ then $H^g \leq C_G(U)$ for every $g \in G$. The definition of G_0 gives $U \leq Z(G_0)$, whence it follows $H^* \leq Z(G_0)$.

ii) We have $a = bh$ for some $h \in H$. Let $\alpha \in A$ be arbitrary and $\beta \in B \cap \alpha H$. Hence $\alpha = \beta h^*$ with $h^* \in H$. Using $HZ^* \trianglelefteq G_0$ and $[A, B] \leq H$ we get $a^\alpha = a^{\beta h^*} = (ah_1)^{h^*} \in aHZ^*$ where $h_1 \in H$. On the other hand, $a^\alpha = (bh)^\alpha = bh_2 h^\alpha$ where $h_2 \in H$, whence it follows $a^\alpha = ah^{-1}h_2 h^\alpha \in aHZ^*$, which means $h^\alpha \in HZ^*$ for every $\alpha \in A$. Using $AH = G$ we can conclude $h \in H \cap \mathrm{core}_G(Z^* H)$. Thus $h \in H^*$.

iii) Let $a \in A \cap G_0$, $\beta \in B$. By $[A, B] \leq H$ we have $a^\beta = ah$ with $h \in H$. Let $\beta^* \in B$ be arbitrary. Clearly $\beta\beta^* = \beta_0 h_0$ with $\beta_0 \in B$, $h_0 \in H$. Then $a^{\beta\beta^*} = (ah)^{\beta^*} = ah_1 h^{\beta *}$ where $h_1 \in H$. So $a^{\beta_0} = a^{h_0^{-1}} h_1 h^{\beta^* h_0^{-1}} \in aH$. Since $HZ^* \trianglelefteq G_0$ we have $a^{h_0^{-1}} \in aHZ^*$ whence $h^{\beta^* h_0^{-1}} \in HZ^*$ i.e. $h^{\beta^*} \in HZ^*$ for every $\beta^* \in B$. Using $BH = G$ it follows $h \in H \cap \mathrm{core}_G(HZ^*)$ i.e. $h \in H^*$. In a similar way we get $[B \cap G_0, A] \leq H^*$.

iv) Let $a_1, a_2 \in A \cap G_0$. We have $a_1 a_2 = ah$ with $a \in A \cap G_0$ and $h \in H$. Let $\beta \in B$ be arbitrary. Using iii) and i) we get $(a_1 a_2)^\beta = a_1 a_2 h_0$ for some $h_0 \in H^*$. As $a^\beta \in aH^*$ we can conclude $(a^{-1} a_1 a_2)^\beta = h^\beta \in H$ for every $\beta \in B$. Since $BH = G$ and $L_G(H) = 1$ we get $h = e$ and $a_1 a_2 \in A$. As G is finite it follows $A \cap G_0 \leq G_0$. In a similar way we can show $B \cap G_0 \leq G_0$.

Let $b_2 \in B \cap a_2 H$. By ii) $a_2 = b_2 h_2$ where $h_2 \in H^*$. Using iii) ${a_1}^{b_2} = a_1 h_1$ where $h_1 \in H^*$. Since $H^* \leq Z(G_0)$ we have ${a_1}^{a_2} = a_1 h_1 \in A \cap G_0$, whence it follows $h_1 = e$ and $A \cap G_0$ is an abelian subgroup. Similarly we can show $B \cap G_0$ is an abelian subgroup, too.

v) By i) and iv) $(A \cap G_0)H^*$ and $(B \cap G_0)H^*$ are abelian subgroups of G. Using ii) it follows $(A \cap G_0)H^* = (B \cap G_0)H^*$. By Lemma 1.2.4 $Z(G) \subseteq A \cap B$, hence $Z^* \subseteq A \cap B \cap G_0$, consequently $U = Z^* H^* \leq (A \cap G_0)H^* = (B \cap G_0)H^* = (A \cap G_0)U = (B \cap G_0)U$. We show $(A \cap G_0)U \trianglelefteq G$. Let $a \in A \cap G_0$. Clearly $aU = bU$ for some $b \in B \cap G_0$. Let $\alpha \in A$, $\beta \in B$ be arbitrary. By using iii) we get $(aU)^\beta = aU$, $(aU)^\alpha = (bU)^\alpha = bU = aU$. Since $\langle A, B \rangle = G$ we can conclude $(aU)^g = aU$ for every $g \in G$, consequently $(A \cap G_0)U$ is normal in G.

vi) Since $G_0/(A \cap G_0)H^* \cong H/H^*$ and H is abelian we get ${G_0}' \leq (A \cap G_0)H^*$. We have $Z^* H \trianglelefteq G_0$, applying Proposition 1.2.8 i) for $G_1 = G_0$ it follows $G_0/Z^* H$ is abelian too, whence ${G_0}' \leq Z^* H \cap (A \cap G_0)H^*$. Using v), we can conclude ${G_0}' \leq Z^* H^* \leq (A \cap G_0)H^*$.

vii) Clearly $Z(G_0) \cap H \leq \mathrm{core}_{G_0} H$.

Let $b \in B$ be arbitrary, then $H^{b^{-1}} \leq G_0$. Let $h_1, h_2 \in H$. We have $G_0 = (A \cap G_0)H$ and $(A \cap G_0)H^* \trianglelefteq G$, hence we get that there exist $a_i, a_j \in A \cap G_0$ and ${h_1}^*, {h_2}^* \in H^*$ such that $a_i h_1 {h_1}^* \in H^{b^{-1}}$ and $a_j h_2 {h_2}^* \in H^{b^{-1}}$. Since $H^* \leq Z(G_0)$ (see i)) and $H^{b^{-1}}$ is abelian, it follows $a_i h_1 \in C_G(a_j h_2)$. Using iv) we get $a_i {h_1}^{a_j} = {a_i}^{{h_2}^{-1}} h_1$. Suppose $h_1 \in \mathrm{core}_{G_0} H$, then ${h_1}^{a_j} \in H$ and vi) gives ${h_1}^{a_j} \in h_1 H^*$, consequently ${a_i}^{-1} {a_i}^{{h_2}^{-1}} \in H^*$ for every $h_2 \in H_2$, that means $a_i \in N_G(H)$. We have $(a_i h_1 {h_1}^*)^b \in H$, by iii) ${a_i}^b \in a_i H^*$ i.e. ${a_i}^b \in N_G(H)$ and by the definition of U and H^*, $({h_1}^*)^b \in U = Z^* \times H^* \leq N_G(H)$. So we get ${h_1}^b \in N_G(H)$ for every $b \in B$. $G = BH$ implies $h_1 \in \mathrm{core}_G(N_G(H) \cap G_0)$. As $N_G(H) = Z(G) \times H$ (see Lemma 1.2.4) it follows $N_G(H) \cap G_0 = (Z(G) \cap G_0) \times H$, and Proposition 1.2.9 iv) gives our statement. □

Now we introduce the following

Definition. Let $\mathcal{F}$ be the class of all pairs (G, H) with the following properties: G is a finite group, H is an abelian subgroup of G, there exist A and B H-connected left transversals to H in G and $\langle A, B\rangle = G$. Let $\mathcal{F}^* \subseteq \mathcal{F}$, we say $\mathcal{F}^*$ is a nice subclass of $\mathcal{F}$, if for every $(G, H) \in \mathcal{F}^*$, $(G/N, HN/N) \in \mathcal{F}$ with some normal subgroup N of G implies $(G/N, HN/N) \in \mathcal{F}^*$.

Proposition 1.2.11 [Cs4, Proposition 3.4]**.** *Let $\mathcal{F}^* \subseteq \mathcal{F}$ be some nice subclass of $\mathcal{F}$. Suppose the group G is of minimal order such that $(G, H) \in \mathcal{F}^*$ and $G' \not\leq N_G(H)$. Denote G_0 the normal closure of H in G.*

Then the following statements are true:

i) *$L_G(H) = 1$ and every minimal normal subgroup of G is elementary abelian of prime power order.*

ii) *If $N \neq 1$ is a normal subgroup of G in G_0, then $G_0 \leq N_G(NH)$.*

iii) *If K_1 and K_2 are minimal normal subgroups of G in G_0, then K_1 and K_2 are elementary abelian subgroups of prime power order for the same prime.*

iv) *$G_0 \cap Z(G)$ is a cyclic subgroup of order p^k for some prime p.*

v) *Denote Z_0 the minimal subgroup of the cyclic group $Z(G) \cap G_0$. Then H and Z_0H are p-subgroups and $Z_0H \trianglelefteq G_0$, furthermore $Z_0H \neq G_0$.*

vi) *G_0 is a p-subgroup.*

vii) *Denote $U = \mathrm{core}_G Z_0H$ and $H_0 = U \cap H$. Then H_0 is elementary abelian p-subgroup, $U = H_0 \times Z_0 \leq Z(G_0)$ and ${G_0}' \leq U$.*

viii) *$H(Z(G) \cap G_0) \trianglelefteq G_0$ and $G_0/H(Z(G) \cap G_0)$ are elementary abelian p-groups.*

ix) *$H/Z(G_0) \cap H$ is elementary abelian p-group.*

x) *$G_0/Z(G_0)$ is an elementary abelian p-group.*

xi) *$G_0/H(Z(G) \cap G_0) \cong H/H \cap Z(G_0)$.*

xii) *There exists no $a \in A$ such that $H^a(H(Z(G) \cap G_0)) = G_0$.*

Proof. i) First we show $L_G(H) = 1$. Suppose $T_0 = L_G(H) \neq 1$, apply Lemma 1.2.7 for $L = L_G(T_0H)$. Then using $L = T_0$ we get $(G/T_0, H/T_0) \in \mathcal{F}$, whence $(G/T_0, H/T_0) \in \mathcal{F}^*$. The minimality of G implies $(G/T_0)' \leq N_{G/T_0}(H/T_0)$, consequently $G' \leq N_G(H)$, a contradiction.

Using Proposition 1.2.8 iii) we get $G_0 \neq G$. By Theorem 1.2.6 G is solvable, whence it follows every minimal normal subgroup of G is elementary abelian of prime power order.

From this point in the proof we can apply Proposition 1.2.9 and Proposition 1.2.10 because $L_G(H) = 1$.

ii) Since $L_G(H) = 1$, then $N \not\leq H$. Put $L = L_G(HN)$. By Lemma 1.2.7 $(G/L, HL/L) \in \mathcal{F}$, whence $(G/L, HL/L) \in \mathcal{F}^*$. The minimality of G gives $(G/L)' \leq N_{G/L}(HL/L)$, whence $HL/L \trianglelefteq G'L/L$. Since $HL = HN$ and $G_0 = G'L$ (see Proposition 1.2.8 ii)) we get our statement.

iii) By Theorem 1.2.6 G is solvable. Suppose $|K_1| = {q_1}^{k_1}$, $|K_2| = {q_2}^{k_2}$ and q_1, q_2 are different primes. By ii), $K_1H \trianglelefteq G_0$ and $K_2H \trianglelefteq G_0$ whence it follows $K_1H \cap K_2H = H \trianglelefteq G_0$, which implies $G' \leq N_G(H)$ because $G_0 = G'H$, a contradiction.

iv) By Proposition 1.2.9 iii) we have $G_0 \cap Z(G) \neq 1$. Suppose $G_0 \cap Z(G)$ is not a cyclic subgroup of prime power order. Then there exist subgroups Z_1 and Z_2 of prime orders in $Z(G) \cap G_0$ such that $Z_1 \neq Z_2$. Using ii), we get $Z_1H \trianglelefteq G_0$, $Z_2H \trianglelefteq G_0$, whence it follows $Z_1H \cap Z_2H = H \trianglelefteq G_0$. Since $G_0 = G'H$ (see Proposition 1.2.8 ii)), then $G' \leq N_G(H)$, a contradiction.

v) We have $G_0 \leq N_G(Z_0H)$ by ii). $Z_0H \neq G_0$, otherwise $G' \leq G_0 \leq N_G(H)$, a contradiction. Assume there is $R \in \mathrm{Syl}_r(H)$ with $r \neq p$. Applying Frattini argument it follows $G_0 = (Z_0H)N_{G_0}(R)$. Since $Z_0H \leq C_G(R)$ then $R \trianglelefteq G_0$. Suppose $R \in \mathrm{Syl}_r(G_0)$. As $G_0 \trianglelefteq G$, Frattini argument implies $R \trianglelefteq G$, which is a contradiction with $L_G(H) = 1$. Thus there exists $R_1 \in \mathrm{Syl}_r(G_0)$ such that $R_1 > R$. We have $G = G_0N_G(R_1)$. Denote S the normal closure of R in G. $R \trianglelefteq G_0$ implies $S \leq R_1$. Let S_0 be a minimal normal subgroup of G in S. As S_0 is an r-subgroup and $r \neq p$ we get a contradiction with iii).

vi) We have Z_0H is a normal p-subgroup of G_0 (see v)). Using Proposition 1.2.10 vi) for $Z^* = Z_0$ we get G_0/Z_0H is abelian. Since $G_0 \neq HZ_0$ by v), there exists $g_1 \in G$ such that $H^{g_1} \not\leq HZ_0$. Then using ${G_0}' \leq Z_0H$ and $H^{g_1} \leq G_0$, we get $H^{g_1}(HZ_0)$ is a normal p-subgroup of G_0. Assume there exists $g_2 \in G$ such that $H^{g_2} \not\leq H^{g_1}(HZ_0)$, then $H^{g_2}H^{g_1}(HZ_0)$ is a normal p-subgroup of G_0, too. Continuing this process, using that G is finite and G_0 is the normal closure of H in G, we can conclude G_0 is a p-subgroup, too.

vii) Suppose there exists $h_0 \in H_0$ such that ${h_0}^p \neq e$. Then using $Z_0H_0 \trianglelefteq G$ and $|Z_0| = p$ we get $({h_0}^p)^g \in H_0$ for all $g \in G$, contradicting $L_G(H) = 1$. By the definition of U, clearly $U = H_0 \times Z_0$. Using Proposition 1.2.10 i) for $Z^* = Z_0$ we conclude $U \leq Z(G_0)$. Proposition 1.2.10 vi) for $Z^* = Z_0$ implies ${G_0}' \leq U$.

viii) $Z(G) \cap G_0$ is a normal subgroup of G, whence ii) implies $H(Z(G) \cap G_0) \trianglelefteq G_0$. Let $g \in G_0 \setminus (Z(G) \cap G_0)H$, $h \in H$. As ${G_0}' \leq Z_0H_0$ (see vii)) we get $h^g = hzh_0$ with $z \in Z_0$, $h_0 \in H_0$. Since $Z_0H_0 \leq Z(G_0)$, H_0 is elementary abelian (see vii)) and $|Z_0| = p$ it follows $h^{(g^p)} = hz^p{h_0}^p = h$ i.e. $g^p \in G_0 \cap C_G(H)$. We have $N_G(H) = C_G(H) = H \times Z(G)$ (see Lemma 1.2.4), whence $G_0 \cap C_G(H) = H \times (Z(G) \cap G_0)$. As ${G_0}' \leq Z_0H_0$ by vii) we can conclude $G_0/H(Z(G) \cap G_0)$ is an elementary abelian p-group.

ix) Assume $H/Z(G_0) \cap H$ is not an elementary abelian p-group. Then the Frattini subgroup $\Phi(H/Z(G_0) \cap H) \neq 1$. Let $F_1 \leq H$ be such that $F_1/Z(G_0) \cap H = \Phi(H/Z(G_0) \cap H)$. Let $g \in G$ be arbitrary. Then we have ${F_1}^g/Z(G_0) \cap H^g = \Phi(H^g/Z(G_0) \cap H^g)$ and $\Phi(H^g/Z(G_0) \cap H^g) = \Phi(H^g)(Z(G_0) \cap H^g)/Z(G_0) \cap H^g$. Since $G_0/H(Z(G) \cap G_0)$ is elementary abelian p-subgroup (see viii)) it follows $H^g/H^g \cap H(Z(G) \cap G_0)$ is elementary abelian p-subgroup, too, consequently $\Phi(H^g) \leq H^g \cap H(Z(G) \cap G_0)$. Hence ${F_1}^g \leq (H^g \cap H(Z(G) \cap G_0))(Z(G_0) \cap H^g)$ and using Proposition 1.2.9 iii) we get ${F_1}^g \leq H(Z(G) \cap G_0)$ for every $g \in G$, consequently $F_1 \leq \mathrm{core}_G H(Z(G) \cap G_0)$. Applying Proposition 1.2.9 iv) we can conclude $F_1 \leq Z(G_0) \cap H$, a contradiction.

x) As ${G_0}' \leq Z(G_0)$ (see vii)) it follows $G_0/Z(G_0)$ is an abelian p-group. Apply Proposition 1.2.10 for $Z^* = Z(G) \cap G_0$, then $\mathrm{core}_G(Z^*H) = \mathrm{core}_G(Z(G) \cap G_0)H = Z(G_0)$ (see Proposition 1.2.9 iv)) and $H^* = Z(G_0) \cap H$, furthermore $(A \cap G_0)H^* = (A \cap G_0)(Z(G_0) \cap H)$ is an abelian normal subgroup of G by Proposition 1.2.10 v). Since $H \cap (A \cap G_0)(Z(G_0) \cap H) = Z(G_0) \cap H$ and $G_0 = (A \cap G_0)H$ it follows $G_0/(A \cap G_0)(Z(G_0) \cap H) \cong H/Z(G_0) \cap H$. Denote $G_1 = (A \cap G_0)(Z(G_0) \cap H)$, by using ix) we can conclude G_0/G_1 is elementary abelian p-group. Denote $G_2 = H(Z(G) \cap G_0)$. By viii) G_0/G_2 is elementary abelian. We have $G_1 \cdot G_2 = G$ and $G_1 \cap G_2 = (Z(G) \cap G_0)(Z(G_0) \cap H) = Z(G_0)$ (see Proposition 1.2.9 iii)). Let $g_0 \in G_0$ be arbitrary. Then $g_0 = g_1 g_2$, where $g_1 \in G_1$, $g_2 \in G_2$. Clearly $g_1 g_2 \in g_2 g_1 Z(G_0)$, whence ${g_0}^p \in {g_1}^p {g_2}^p Z(G_0)$. Since $G_0/G_1 \cong G_2/G_1 \cap G_2$ and $G_0/G_2 \cong G_1/G_2 \cap G_1$, furthermore G_0/G_1 and G_0/G_2 are elementary abelian we get ${g_1}^p \in Z(G_0)$, ${g_2}^p \in Z(G_0)$, consequently ${g_0}^p \in Z(G_0)$. Thus $G_0/Z(G_0)$ is an elementary abelian p-group.

xi) Claim 1. We show if $c \notin N_G(H)$, then $H \cap H^c$ is a maximal subgroup of H.

Proof of Claim 1. We have by v) $Z_0 H \trianglelefteq G_0$, $|Z_0| = p$ whence $|H^c : H^c \cap H| = p$, which implies our statement. □

Apply Proposition 1.2.10 for $Z^* = Z(G) \cap G_0$. Then $H^* = (\mathrm{core}_G(Z(G) \cap G_0)H) \cap H = Z(G_0) \cap H$ (see Proposition 1.2.9 iv)) and by Proposition 1.2.10 v) we get $(A \cap G_0)(Z(G_0) \cap H)$ is an abelian normal subgroup of G. Proposition 1.2.10 vi) gives ${G_0}' \leq (Z(G) \cap G_0)(Z(G_0) \cap H) = Z(G_0) \leq (A \cap G_0)(Z(G_0) \cap H)$ (by Lemma 1.2.4 $Z(G) \cap G_0 \subseteq A \cap G_0$).

Denote $W_1 = (A \cap G_0)(Z(G_0) \cap H)$.

Using $G_0 = (A \cap G_0)HZ(G_0) = (A \cap G_0)H(Z(G_0) \cap H)(Z(G) \cap G_0)$, $(A \cap G_0)(Z(G_0) \cap H) \cap H(Z(G) \cap G_0) = (Z(G_0) \cap H)(Z(G) \cap G_0) = Z(G_0)$ we get $G_0 \ / \ H(Z(G) \cap G_0) \cong W_1 \ / \ Z(G_0)$. Denote $\overline{W}_1 = W_1/Z(G_0)$. We have $\overline{W}_1$ is an elementary abelian p-group by viii).

Claim 2. If $h \in H \setminus (H \cap Z(G_0))$ and $c \in W_1$, then $h^{-1}h^c \in Z_0(H \cap Z(G_0))$.

Proof of Claim 2. Since ${G_0}' \le Z(G_0)$ (see Proposition 1.2.10 vi)) it follows $h^{-1}h^c \in Z(G_0) \cap HZ_0$. $Z(G_0) = (Z(G) \cap G_0)(Z(G_0) \cap H)$ implies our statement. $\square$

Denote $W_2 = HZ_0$. By v) we have $W_2 \trianglelefteq G_0$.

Denote $\overline{W}_2 = W_2/H \cap Z(G_0)$ and $\overline{H} = H/H \cap Z(G_0)$.

By ix) $\overline{H}$ is an elementary abelian p-group. The elements of $\overline{W}_1$ induce automorphisms on the elements of $\overline{W}_2$. Denote $C_{\overline{W}_1}(\overline{H})$ those elements of $\overline{W}_1$ which fix every element of $\overline{H}$. By using Claim 2 we can conclude $C_{\overline{W}_1}(\overline{H}) = 1$.

Claim 3. We show the number of minimal subgroups of $\overline{H}$ is smaller than or equal to the number of maximal subgroups of $\overline{W}_1$.

Proof of Claim 3. Let $\overline{d} \in \overline{H}$. Since $\mathrm{core}_{G_0} H = Z(G_0) \cap H$ (see Proposition 1.2.10 vii)) it follows $C_{\overline{W}_1}(\overline{d}) \neq \overline{W}_1$. Using Claim 2 and $|Z_0| = p$ we can conclude $C_{\overline{W}_1}(\overline{d})$ is a maximal subgroup of $\overline{W}_1$. We show that if $\overline{d}_1 \in \overline{H} \setminus \langle \overline{d} \rangle$, then $C_{\overline{W}_1}(\overline{d}_1) \neq C_{\overline{W}_1}(\overline{d})$. Suppose $C_{\overline{W}_1}(\overline{d}_1) = C_{\overline{W}_1}(\overline{d})$. Let $\overline{c}_0 \in \overline{W}_1 \setminus C_{\overline{W}_1}(\overline{d})$, and let $c_0 \in W_1$, $d, d_1 \in H$ be such that $\overline{c}_0 = c_0 Z(G_0)$, $\overline{d} = d(H \cap Z(G_0))$, $\overline{d}_1 = d_1(H \cap Z(G_0))$. Clearly $c_0 \notin N_G(H)$, $d, d_1 \notin H \cap Z(G_0)$ and $\langle d, d_1 \rangle^{c_0} \le H^{c_0}$. We have $|H^{c_0} : H^{c_0} \cap H| = p$ and $|\langle d, d_1 \rangle| \ge p^2$, hence there exists $1 \neq d_2 \in \langle d, d_1 \rangle$ such that ${d_2}^{c_0} \in H$. By Claim 2 $\overline{d}_2^{\,\overline{c}_0} = \overline{d}_2$. We have $C_{\overline{W}_1}(\overline{d}) \le C_{\overline{W}_1}(\overline{d}_2)$. The maximality of $C_{\overline{W}_1}(\overline{d})$ gives $\overline{W}_1 = \left\langle C_{\overline{W}_1}(\overline{d}), \overline{c}_0 \right\rangle$, consequently $C_{\overline{W}_1}(\overline{d}_2) = \overline{W}_1$, which is impossible by $\mathrm{core}_{G_0} H = Z(G_0) \cap H$ (see Proposition 1.2.10 vii)). $\square$

Claim 4. We show the number of minimal subgroups of $\overline{W}_1$ is smaller than or equal to the number of maximal subgroups of $\overline{H}$.

Proof of Claim 4. Let $\overline{c} \in \overline{W}_1$ and $c \in W_1$ such that $\overline{c} = cZ(G_0)$. Since

$$\overline{W}_1 = (A \cap G_0)(Z(G_0) \cap H)/Z(G_0) \text{ and } N_G(H) \cap G_0 = H \times (Z(G) \cap G_0)$$

(see Lemma 1.2.4 and Proposition 1.2.9 iii)) it follows $c \notin N_G(H)$.

Using Claim 1 we can conclude $\overline{H} \cap \overline{H}^{\overline{c}}$ is a maximal subgroup of $\overline{H}$. We show that if $\overline{c}_1 \in \overline{W}_1$ and $1 \neq \overline{c}_1 \notin \langle \overline{c} \rangle$, then $\overline{H} \cap \overline{H}^{\overline{c}_1} \neq \overline{H} \cap \overline{H}^{\overline{c}}$. Suppose $\overline{H} \cap \overline{H}^{\overline{c}_1} = \overline{H} \cap \overline{H}^{\overline{c}}$. Let $\overline{h}_0 \in \overline{H} \setminus (\overline{H} \cap \overline{H}^{\overline{c}})$, let $c_1 \in W_1$ be such that $\overline{c}_1 = c_1 Z(G_0)$. Clearly $c_1 \notin N_G(H)$, hence by Claim 2 ${h_0}^{-1}{h_0}^{c_1} \in Z_0(H \cap Z(G_0))$ and ${h_0}^{-1}{h_0}^{c} \in Z_0(H \cap Z(G_0))$. Using $|Z_0| = p$ there exists $1 \le j \le p-1$ such that $\overline{h}_0^{\,\overline{c}^j} = \overline{h}_0^{\,\overline{c}_1}$ i.e. $1 \neq \overline{c}^j {\overline{c}_1}^{-1} \in C_{\overline{W}_1}(\overline{h}_0)$. By Claim 2 $\overline{c}_1, \overline{c} \in C_{\overline{W}_1}(\overline{H} \cap \overline{H}^{\overline{c}})$. The maximality of $\overline{H} \cap \overline{H}^{\overline{c}}$ in $\overline{H}$ implies $\langle \overline{H} \cap \overline{H}^{c}, \overline{h}_0 \rangle = \overline{H}$ consequently $1 \neq \overline{c}^j {\overline{c}_1}^{-1} \in C_{\overline{W}_1}(\overline{H})$. By Claim 2 we have $C_{\overline{W}_1}(\overline{H}) = 1$. A contradiction. $\square$

The number of minimal subgroups of a finite elementary abelian p-group is equal to the number of its maximal subgroups by duality, whence Claim 3 and Claim 4 imply our statement.

xii) Denote $H_1 = H \cap Z(G_0)$ and $Z_1 = Z(G) \cap G_0$. Hence by Proposition 1.2.9 iii), $Z(G_0) = Z_1 \times H_1$. Assume there exists $a \in A$ such that $H^a \cdot Z_1 H = G_0$. Using viii), ix), and xi) it follows that $H \cap H^a \le H_1$ and $H^a \cap HZ_1 \le H_1 Z_1$.

1) We show $h = a^{-1}b \in H_1$ where $b \in B \cap aH$.

By $[A, B] \le H$ we have $h^a \in H \cap H^a \le H_1$. Since $H_1Z_1 \trianglelefteq G$ then $h \in H_1$.

2) We show $H \cap H^b \le H_1$ and $H^b \cap HZ_1 \le H_1Z_1$.

Using $a^{-1}b \in H_1$, $H \cap H^a \le H_1$ and $H^a \cap HZ_1 \le H_1Z_1$ it is trivial.

3) We claim $a_0^{-1}b_0 \in H_1$ for every $a_0 \in A$, $b_0 \in B \cap a_0H$.

Let $a_0 = b_0h_0$ with $h_0 \in H$. Using $[A, B] \le H$ we have $b^{a_0} = (ah)^{b_0h_0} = (awh^{b_0})^{h_0} \in a^{h_0}wZ(G_0)$ where $w \in H$. Since $b^{a_0} \in bH = aH$ and $Z(G_0) = H_1 \times Z_1$ it follows $a^{h_0} \in aHZ_1$, consequently ${h_0}^a \in HZ_1$. Using $H^a \cap HZ_1 \le H_1Z_1$ and $H_1Z_1 \trianglelefteq G$ we can conclude $h_0 \in H_1$.

4) We show $a^{b_0} \in aH_1$ for every $b_0 \in B$.

Let $a^{b_0} = aw$ with $w \in H$ as in 3).

Suppose $\mathcal{O}(a) = 2$. Then using $[A, B] \le H$, $(a^2)^{b_0} = (aw)^2 = a^2w^aw = e$, consequently $w^a = w^{-1}$ which means $w^a \in H \cap H^a \le H_1$. Hence $H_1Z_1 \trianglelefteq G$ gives $w \in H_1$.

Suppose $\mathcal{O}(a) \ne 2$. Then let $a^2 = b_1h^*$ where $b_1 \in B$ and $h^* \in H$. Clearly $a^2 = (b_1h^*)^a = b_1h_2(h^*)^a \in b_1H$ where $h_2 \in H$, which implies $(h^*)^a \in H \cap H^a$. Since $H \cap H^a \le H_1$ and $H_1Z_1 \trianglelefteq G$ we get $h^* \in H_1$. Let $a_0 \in A$, $b_0 \in B \cap a_0H$. By 3) we have $a_0 = b_0h_0$ with $h_0 \in H_1$. Denote $u = (h^*)^{-1}$. Using $[A, B] \le H$ and $H_1Z_1 \trianglelefteq G$ we get $b_1^{a_0} = (a^2u)^{b_0h_0} \in ((aw)^2)^{h_0}H_1Z_1$. Hence $b_1^{a_0} \in (a^2)^{h_0}(w^a)^{h_0}HZ_1$ holds. Since $h_0 \in H_1$ and $H_1Z_1 \trianglelefteq G$ we have $b_1^{a_0} \in a^2(w^a)^{h_0}HZ_1$. On the other hand, $b_1^{a_0} \in a^2H$, thus we can conclude $(w^a)^{h_0} \in HZ_1$, whence $w^a \in H^a \cap HZ_1$. Since $H^a \cap HZ_1 \le H_1Z_1$ using $H_1Z_1 \trianglelefteq G$ it follows $w \in H_1$.

5) We claim $b^{a_0} \in bH_1$ for every $a_0 \in A$.

Using 1) and 2) we can show it similarly to 4).

6) We show $AH_1 \le G$, $BH_1 \le G$.

Let $a_1, a_2 \in A$, we have $a_1a_2 = a_3t$ with $a_3 \in A$, $t \in H$. Using 5) $(a_1a_2)^b = a_1h_1a_2h_2$, where $h_1, h_2 \in H_1$. Since $H_1Z_1 \trianglelefteq G$ easily $(a_1a_2)^b \in a_1a_2H_1Z_1$. As ${a_3}^b \in a_3H_1$ (see 5)) it follows $t^b \in H^b \cap HZ_1$, consequently $t^b \in H_1Z_1$ by 2). So $H_1Z_1 \trianglelefteq G$ implies $t \in H_1$. The finiteness of G gives $\langle A\rangle \subseteq AH_1$. In a similar way we can show $\langle B\rangle \subseteq BH_1$. Hence using $H_1Z_1 \trianglelefteq G$ we can conclude $AH_1Z_1 \le G$, $BH_1Z_1 \le G$. Applying Proposition 1.2.9 v) we get $AH_1 \le G$, $BH_1 \le G$.

Using 3) it follows $AH_1 = BH_1$. Since $\langle A, B\rangle = G$, then $H = H_1$. As $H_1 \le Z(G_0)$ and $G' \le G_0$ (see Proposition 1.2.8 ii)) we can conclude $G' \le N_G(H)$. This is the final contradiction. $\square$

Let Z_0 and H_0 be as in Proposition 1.2.11.

Corollary 1.2.12 [Cs5, Corollary 3.4]**.** *Assume $Z_0 = Z(G) \cap G_0$. Then the following statements are true:*

i) *$Z(G_0) \cap H = H_0$, $Z(G_0) = Z_0 \times H_0$.*

ii) H/H_0 *is elementary abelian* p*-group.*

iii) $G_0/HZ_0 \cong H/H_0$.

iv) $G_0/H_0 \times Z_0$ *is elementary abelian* p*-group.*

Proof. i) By Proposition 1.2.9 iv) $\mathrm{core}_G Z_0 H = Z(G_0)$, consequently $Z(G_0) \cap H = H_0$. From Proposition 1.2.9 iii) it follows $Z(G_0) = Z_0 \times H_0$.

ii) It is obvious by i) and Proposition 1.2.11 ix).

iii) By Proposition 1.2.11 xi) it is trivial.

iv) See i) and Proposition 1.2.11 x). $\square$

Answer to Problem 1.2.1*

Construction of the counterexample

We construct a finite group G with the following properties: there is an abelian subgroup H, there exist A and B H-connected left transversals to H in G such that $\langle A, B\rangle = G$, $\mathrm{core}_G H = 1$ and $G' \not\leq N_G(H)$.

Taking into consideration our previous propositions we choose some parameters.

Let $p = 2$, $|H_0| = 2^3$ and $|H| = 2^6$.

Supposing $Z_0 = G_0 \cap Z(G)$ it follows $|G_0 \cap Z(G)| = 2$ (see Proposition 1.2.11 iv)) and using Corollary 1.2.12 ii) we have H/H_0 is elementary abelian of order 2^3. From Corollary 1.2.12 iii) we can conclude $|G_0 : HZ_0| = 2^3$, consequently $|G_0| = 2^{10}$ holds.

Let $H_0 = \langle h_1, h_2, h_3\rangle$, $T = \langle {h_1}^*, {h_2}^*, {h_3}^*\rangle$ be elementary abelian 2-groups of order 2^3. Denote $H = H_0 \times T$. Let $Z_0 = \langle z\rangle$ be of order 2. Let $K_0 = \langle a_1, a_2, a_3\rangle$ be an elementary abelian group of order 2^3. Define G_0 as the semidirect product of $H \times \langle z\rangle$ by K_0, i.e. $G_0 = (H \times \langle z\rangle) \rtimes K_0$ in the following way:

$$z^{a_i} = z \quad {h_i}^{a_j} = h_i \quad ({h_i}^*)^{a_i} = {h_i}^* z \quad \text{for every } 1 \leq i, j \leq 3$$
$$\text{and } ({h_i}^*)^{a_j} = {h_i}^* \text{ for every } 1 \leq i, j \leq 3, \;\; i \neq j.$$

Obviously G_0 is really a semidirect product of order 2^{10}.

The structure of our G_0 has the properties of the normal closure G_0 of H in the minimal counterexample (supposing $|Z_0| = p$) described in Proposition 1.2.11. Indeed, H/H_0 and $G_0/HZ_0 \cong K_0$ are elementary abelian 2-groups, furthermore $|G_0/HZ_0| = |H/H_0| = 2^3$. We can see $Z(G_0) = Z_0 \times H_0$ and $G_0/Z(G_0) \cong K_0 T$ is elementary abelian 2-group, too.

We construct a 2-group G such that G/G_0 is elementary abelian of order 2^3.

Let $T_1 = \langle \gamma_1\rangle$ be of order 2. Define M_1 as the semidirect product of G_0 by T_1, i.e. $M_1 = G_0 \rtimes T_1$ in the following way:

$$\begin{array}{llll} & {h_1}^{\gamma_1} = h_1 z & ({h_1}^*)^{\gamma_1} = {h_1}^* & {a_1}^{\gamma_1} = a_1 z \\ z^{\gamma_1} = z & {h_2}^{\gamma_1} = h_2 & ({h_2}^*)^{\gamma_1} = a_3 {h_2}^* h_3 & {a_2}^{\gamma_1} = a_2 \\ & {h_3}^{\gamma_1} = h_3 & ({h_3}^*)^{\gamma_1} = a_2 {h_3}^* h_2 & {a_3}^{\gamma_1} = a_3. \end{array}$$

Using the structure of G_0 it is clear that for the proof that M_1 is really a semidirect product it is sufficient to show the following equalities and the reader can check them easily:

a) ${a_i}^{\gamma_1} {h_j}^{\gamma_1} {a_i}^{\gamma_1} = {h_j}^{\gamma_1}$ for every $1 \leq i, j \leq 3$

b) ${a_i}^{\gamma_1} ({h_i}^*)^{\gamma_1} {a_i}^{\gamma_1} = {h_i}^{*\gamma_1} z^{\gamma_1}$ for every $1 \leq i \leq 3$

c) ${a_i}^{\gamma_1} ({h_j}^*)^{\gamma_1} {a_i}^{\gamma_1} = {h_j}^{*\gamma_1}$ for every $1 \leq i, j \leq 3$, $i \neq j$

d) ${h_i}^{({\gamma_1}^2)} = h_i$, ${a_i}^{({\gamma_1}^2)} = a_i$, $({h_i}^*)^{({\gamma_1}^2)} = {h_i}^*$ for every $1 \leq i \leq 3$.

Let $T_2 = \langle \gamma_2 \rangle$ be of order 2. Define M_2 as the semidirect product of M_1 by T_2, i.e. $M_2 = M_1 \rtimes T_2$ in the following way:

$$\begin{array}{llll} & {h_1}^{\gamma_2} = h_1 & ({h_1}^*)^{\gamma_2} = a_3{h_1}^* & {a_1}^{\gamma_2} = a_1 \\ z^{\gamma_2} = z & {h_2}^{\gamma_2} = h_2z & ({h_2}^*)^{\gamma_2} = {h_2}^* & {a_2}^{\gamma_2} = a_2z \\ & {h_3}^{\gamma_2} = h_3 & ({h_3}^*)^{\gamma_2} = a_1{h_3}^* & {a_3}^{\gamma_2} = a_3 \end{array}$$

and
$$ {\gamma_1}^{\gamma_2} = \gamma_1 a_3 h_3. $$

The structure of M_1 shows that for the proof that M_2 is really a semidirect product it is sufficient to verify the following equalities and one can do it with not too big difficulties:

a) ${a_i}^{\gamma_2}{h_j}^{\gamma_2}{a_i}^{\gamma_2} = {h_j}^{\gamma_2}$ for every $1 \le i,j \le 3$
b) ${a_i}^{\gamma_2}({h_i}^*)^{\gamma_2}{a_i}^{\gamma_2} = ({h_i}^*)^{\gamma_2}z^{\gamma_2}$ for every $1 \le i \le 3$
c) ${a_i}^{\gamma_2}({h_j}^*)^{\gamma_2}{a_i}^{\gamma_2} = ({h_j}^*)^{\gamma_2}$ for every $1 \le i,j \le 3,\ \ i \ne j$
d) ${\gamma_1}^{\gamma_2}{h_1}^{\gamma_2}{\gamma_1}^{\gamma_2} = {h_1}^{\gamma_2}z^{\gamma_2}$
e) ${\gamma_1}^{\gamma_2}{h_i}^{\gamma_2}{\gamma_1}^{\gamma_2} = {h_i}^{\gamma_2}$ $i = 2,3$
f) ${\gamma_1}^{\gamma_2}{a_1}^{\gamma_2}{\gamma_1}^{\gamma_2} = {a_1}^{\gamma_2}z^{\gamma_2}$
g) ${\gamma_1}^{\gamma_2}{a_i}^{\gamma_2}{\gamma_1}^{\gamma_2} = {a_i}^{\gamma_2}$ $i = 2,3$
h) ${\gamma_1}^{\gamma_2}({h_1}^*)^{\gamma_2}{\gamma_1}^{\gamma_2} = ({h_1}^*)^{\gamma_2}$
i) ${\gamma_1}^{\gamma_2}({h_2}^*)^{\gamma_2}{\gamma_1}^{\gamma_2} = {a_3}^{\gamma_2}({h_2}^*)^{\gamma_2}{h_3}^{\gamma_2}$
j) ${\gamma_1}^{\gamma_2}({h_3}^*)^{\gamma_2}{\gamma_1}^{\gamma_2} = {a_2}^{\gamma_2}({h_3}^*)^{\gamma_2}{h_2}^{\gamma_2}$
k) ${h_i}^{({\gamma_2}^2)} = h_i,\ \ ({h_i}^*)^{({\gamma_2}^2)} = {h_i}^*,\ \ {a_i}^{({\gamma_2}^2)} = a_i$ for every $1 \le i \le 3$
l) ${\gamma_1}^{({\gamma_2}^2)} = \gamma_1$

Let $T_3 = \langle \gamma_3 \rangle$ be again a group of order two. Define G as the semidirect product of M_2 by T_3, i.e. $G = M_2 \rtimes T_3$ in the following way:

$$\begin{array}{lll} & {h_1}^{\gamma_3} = h_1 & ({h_1}^*)^{\gamma_3} = a_2{h_1}^*h_2 \\ z^{\gamma_3} = z & {h_2}^{\gamma_3} = h_2 & ({h_2}^*)^{\gamma_3} = a_1{h_2}^* \\ & {h_3}^{\gamma_3} = h_3z & ({h_3}^*)^{\gamma_3} = {h_3}^* \end{array}$$

$$\begin{array}{ll} {a_1}^{\gamma_3} = a_1 & \\ {a_2}^{\gamma_3} = a_2 & {\gamma_1}^{\gamma_3} = \gamma_1 a_2 h_2 \\ {a_3}^{\gamma_3} = a_3 z & {\gamma_2}^{\gamma_3} = \gamma_2 a_1 h_1. \end{array}$$

Using the structure of M_2 one can conclude that for the proof that G is a semidirect product, it is sufficient to check the following equalities and we leave it for the reader:

a) $a_i^{\gamma_3} h_j^{\gamma_3} a_i^{\gamma_3} = h_j^{\gamma_3}$ for every $1 \leq i, j \leq 3$

b) $a_i^{\gamma_3} (h_i^*)^{\gamma_3} a_i^{\gamma_3} = (h_i^*)^{\gamma_3} z^{\gamma_3}$ for every $1 \leq i \leq 3$

c) $a_i^{\gamma_3} (h_j^*)^{\gamma_3} a_i^{\gamma_3} = (h_j^*)^{\gamma_3}$ for every $1 \leq i, j \leq 3, \;\; i \neq j$

d) $\gamma_1^{\gamma_3} h_1^{\gamma_3} \gamma_1^{\gamma_3} = h_1^{\gamma_3} z^{\gamma_3}$

e) $\gamma_1^{\gamma_3} h_i^{\gamma_3} \gamma_1^{\gamma_3} = h_i^{\gamma_3}$ $i = 2, 3$

f) $\gamma_1^{\gamma_3} a_1^{\gamma_3} \gamma_1^{\gamma_3} = a_1^{\gamma_3} z^{\gamma_3}$

g) $\gamma_1^{\gamma_3} a_i^{\gamma_3} \gamma_1^{\gamma_3} = a_i^{\gamma_3}$ $i = 2, 3$

h) $\gamma_1^{\gamma_3} (h_1^*)^{\gamma_3} \gamma_1^{\gamma_3} = (h_1^*)^{\gamma_3}$

i) $\gamma_1^{\gamma_3} (h_2^*)^{\gamma_3} \gamma_1^{\gamma_3} = a_3^{\gamma_3} (h_2^*)^{\gamma_3} h_3^{\gamma_3}$

j) $\gamma_1^{\gamma_3} (h_3^*)^{\gamma_3} \gamma_1^{\gamma_3} = a_2^{\gamma_3} (h_3^*)^{\gamma_3} h_2^{\gamma_3}$

k) $\gamma_2^{\gamma_3} h_i^{\gamma_3} \gamma_2^{\gamma_3} = h_i^{\gamma_3}$ $i = 1, 3$

l) $\gamma_2^{\gamma_3} h_2^{\gamma_3} \gamma_2^{\gamma_3} = h_2^{\gamma_3} z^{\gamma_3}$

m) $\gamma_2^{\gamma_3} a_i^{\gamma_3} \gamma_2^{\gamma_3} = a_i^{\gamma_3}$ $i = 1, 3$

n) $\gamma_2^{\gamma_3} a_2^{\gamma_3} \gamma_2^{\gamma_3} = a_2^{\gamma_3} z^{\gamma_3}$

o) $\gamma_2^{\gamma_3} (h_1^*)^{\gamma_3} \gamma_2^{\gamma_3} = a_3^{\gamma_3} (h_1^*)^{\gamma_3}$

p) $\gamma_2^{\gamma_3} (h_2^*)^{\gamma_3} \gamma_2^{\gamma_3} = (h_2^*)^{\gamma_3}$

r) $\gamma_2^{\gamma_3} (h_3^*)^{\gamma_3} \gamma_2^{\gamma_3} = a_1^{\gamma_3} (h_3^*)^{\gamma_3}$

s) $\gamma_2^{\gamma_3} \gamma_1^{\gamma_3} \gamma_2^{\gamma_3} = \gamma_1^{\gamma_3} a_3^{\gamma_3} h_3^{\gamma_3}$

t) $h_i^{(\gamma_3^2)} = h_i, \;\; a_i^{(\gamma_3^2)} = a_i, \;\; (h_i^*)^{(\gamma_3^2)} = h_i^*$ for every $1 \leq i \leq 3$

u) $\gamma_1^{(\gamma_3^2)} = \gamma_1, \;\; \gamma_2^{(\gamma_3^2)} = \gamma_2$

Let $b_i = a_i h_i$, $\beta_i = \gamma_i h_i^*$ for every $1 \leq i \leq 3$. Denote $K = K_0 \times \langle z \rangle$.

At the construction of H-connected transversals we paid attention to Proposition 1.2.9.

Let

$$A = \{n_i, \; \gamma_1 n_i, \; \gamma_2 n_i, \; \gamma_3 n_i, \; \gamma_1\gamma_2 n_i, \; \gamma_1\gamma_3 n_i, \; \gamma_2\gamma_3 h_1 n_i, \; \gamma_1\gamma_2\gamma_3 h_1 n_i \; / \; n_i \in K\}.$$

Denote $W = \langle z, b_1, b_2, b_3 \rangle$. By the definition of b_i obviously W is elementary abelian 2-group of order 2^4.

Let

$$B = \{w_i, \; \beta_1 w_i, \; \beta_2 w_i, \; \beta_3 w_i, \; \beta_1\beta_2 w_i, \; \beta_1\beta_3 h_2 w_i, \; \beta_2\beta_3 w_i, \; \beta_1\beta_2\beta_3 h_2 w_i \; / \; w_i \in W\}.$$

Claim 1. *We state A and B are H-connected left transversals to H in G.*

Proof. We have $|G:H| = 2^7$, since $|W| = |K| = 2^4$ it follows $|A| = |B| = 2^7$. The construction of G implies that A and B are left transversals to H in G. Thus we need only to show $[A, B] \leq H$. Obviously we have $z \in Z(G)$.

(1) Clearly $a_i^{b_j} = a_i$ for every $1 \leq i, j \leq 3$.

(2) It can be shown easily:

$$a_i{}^{\beta_j} = a_i \text{ and } b_i{}^{\gamma_j} = b_i \text{ for every } 1 \le i, j \le 3.$$

(3) Clearly $h_i{}^{\beta_i} = h_i z$ and $h_i{}^{\beta_j} = h_i$ for every $1 \le i, j \le 3$, $i \ne j$. Using these relations after some calculation one can obtain the following:

$$\gamma_i{}^{\beta_i} = \gamma_i \text{ for every } 1 \le i \le 3$$

$$\begin{aligned} &\gamma_1{}^{\beta_2} = \gamma_1, \quad \gamma_2{}^{\beta_1} = \gamma_2 h_3 \quad \gamma_3{}^{\beta_1} = \gamma_3, \\ &\gamma_1{}^{\beta_3} = \gamma_1, \quad \gamma_2{}^{\beta_3} = \gamma_2 h_1, \quad \gamma_3{}^{\beta_2} = \gamma_3 h_1. \end{aligned}$$

Thus $\gamma_i{}^{\beta_i} \in \gamma_i H$ for every $1 \le i, j \le 3$.

(4) By applying (3) one can verify easily

$$\begin{array}{lll} (\gamma_1\gamma_2)^{\beta_1} = \gamma_1\gamma_2 h_3 & (\gamma_1\gamma_3)^{\beta_1} = \gamma_1\gamma_3 & (\gamma_2\gamma_3 h_1)^{\beta_1} = \gamma_2\gamma_3 h_1 h_3 \\ (\gamma_1\gamma_2)^{\beta_2} = \gamma_1\gamma_2 & (\gamma_1\gamma_3)^{\beta_2} = \gamma_1\gamma_3 h_1 & (\gamma_2\gamma_3 h_1)^{\beta_2} = \gamma_2\gamma_3 \\ (\gamma_1\gamma_2)^{\beta_3} = \gamma_1\gamma_2 h_1 & (\gamma_1\gamma_3)^{\beta_3} = \gamma_1\gamma_3 & (\gamma_2\gamma_3 h_1)^{\beta_3} = \gamma_2\gamma_3. \end{array}$$

Consequently, $(\gamma_1\gamma_2)^{\beta_i} \in \gamma_1\gamma_2 H$, $(\gamma_1\gamma_3)^{\beta_i} \in \gamma_1\gamma_3 H$ and $(\gamma_2\gamma_3 h_1)^{\beta_i} \in \gamma_2\gamma_3 H$ for every $1 \le i \le 3$.

(5) The reader can check:

$$\begin{aligned} (\gamma_1\gamma_2\gamma_3 h_1)^{\beta_1} &= \gamma_1\gamma_2\gamma_3 h_3 h_1 \\ (\gamma_1\gamma_2\gamma_3 h_1)^{\beta_2} &= \gamma_1\gamma_2\gamma_3 \\ (\gamma_1\gamma_2\gamma_3 h_1)^{\beta_3} &= \gamma_1\gamma_2\gamma_3. \end{aligned}$$

Hence we get $(\gamma_1\gamma_2\gamma_3 h_1)^{\beta_i} \in \gamma_1\gamma_2\gamma_3 H$ for every $1 \le i \le 3$.

(6) Using (3) we can show:

$$\begin{array}{lll} \gamma_1{}^{\beta_1\beta_2} = \gamma_1 & \gamma_1{}^{\beta_1\beta_3 h_2} = \gamma_1 & \gamma_1{}^{\beta_2\beta_3} = \gamma_1 \\ \gamma_2{}^{\beta_1\beta_2} = \gamma_2 h_3 & \gamma_2{}^{\beta_1\beta_3 h_2} = \gamma_2 h_1 h_3 & \gamma_2{}^{\beta_2\beta_3} = \gamma_2 h_1 \\ \gamma_3{}^{\beta_1\beta_2} = \gamma_3 h_1 & \gamma_3{}^{\beta_1\beta_3 h_2} = \gamma_3 & \gamma_3{}^{\beta_2\beta_3} = \gamma_3 h_1. \end{array}$$

Hence it follows $\gamma_i{}^{\beta_1\beta_2} \in \gamma_i H$, $\quad \gamma_i{}^{\beta_1\beta_3 h_2} \in \gamma_i H$, $\quad \gamma_i{}^{\beta_2\beta_3} \in \gamma_i H$ for every $1 \le i \le 3$.

(7) By applying again (3) one can verify:

$$\begin{aligned}(\gamma_1\gamma_2)^{\beta_1\beta_2} &= \gamma_1\gamma_2 h_3 \in \gamma_1\gamma_2 H\\ (\gamma_1\gamma_2)^{\beta_1\beta_3 h_2} &= \gamma_1\gamma_2 h_1 h_2 \in \gamma_1\gamma_2 H\\ (\gamma_1\gamma_2)^{\beta_2\beta_3} &= \gamma_1\gamma_2 h_1 \in \gamma_1\gamma_2 H\end{aligned}$$

$$\begin{aligned}(\gamma_1\gamma_3)^{\beta_1\beta_2} &= \gamma_1\gamma_3 h_1 \in \gamma_1\gamma_3 H\\ (\gamma_1\gamma_3)^{\beta_1\beta_3 h_2} &= \gamma_1\gamma_3 \in \gamma_1\gamma_3 H\\ (\gamma_1\gamma_3)^{\beta_2\beta_3} &= \gamma_1\gamma_3 h_1 \in \gamma_1\gamma_3 H\end{aligned}$$

$$\begin{aligned}(\gamma_2\gamma_3 h_1)^{\beta_1\beta_2} &= \gamma_2\gamma_3 h_3 \in \gamma_2\gamma_3 h_1 H\\ (\gamma_2\gamma_3 h_1)^{\beta_1\beta_3 h_2} &= \gamma_2\gamma_3 h_3 \in \gamma_2\gamma_3 h_1 H\\ (\gamma_2\gamma_3 h_1)^{\beta_2\beta_3} &= \gamma_2\gamma_3 h_1 \in \gamma_2\gamma_3 h_1 H\end{aligned}$$

(8) Using (5) we get:

$$\begin{aligned}(\gamma_1\gamma_2\gamma_3 h_1)^{\beta_1\beta_2} &= \gamma_1\gamma_2\gamma_3 h_3 \in \gamma_1\gamma_2\gamma_3 h_1 H\\ (\gamma_1\gamma_2\gamma_3 h_1)^{\beta_1\beta_3 h_2} &= \gamma_1\gamma_2\gamma_3 h_3 \in \gamma_1\gamma_2\gamma_3 h_1 H\\ (\gamma_1\gamma_2\gamma_3 h_1)^{\beta_2\beta_3} &= \gamma_1\gamma_2\gamma_3 h_1 \in \gamma_1\gamma_2\gamma_3 h_1 H.\end{aligned}$$

(9) One can check:

$$\begin{aligned}{\gamma_1}^{\beta_1\beta_2\beta_3 h_2} &= \gamma_1 \in \gamma_1 H\\ {\gamma_2}^{\beta_1\beta_2\beta_3 h_2} &= \gamma_2 h_1 h_3 \in \gamma_2 H\\ {\gamma_3}^{\beta_1\beta_2\beta_3 h_2} &= \gamma_3 h_1 \in \gamma_3 H.\end{aligned}$$

(10) From (3) and (4) we can conclude:

$$\begin{aligned}(\gamma_1\gamma_2)^{\beta_1\beta_2\beta_3 h_2} &= \gamma_1\gamma_2 h_1 h_3 \in \gamma_1\gamma_2 H\\ (\gamma_1\gamma_3)^{\beta_1\beta_2\beta_3 h_2} &= \gamma_1\gamma_3 h_1 \in \gamma_1\gamma_3 H\\ (\gamma_2\gamma_3 h_1)^{\beta_1\beta_2\beta_3 h_2} &= \gamma_2\gamma_3 h_1 h_3 \in \gamma_2\gamma_3 h_1 H.\end{aligned}$$

(11) Using (5) and (3) after some calculation we get:

$$(\gamma_1\gamma_2\gamma_3 h_1)^{\beta_1\beta_2\beta_3 h_2} = \gamma_1\gamma_2\gamma_3 h_1 h_3 \in \gamma_1\gamma_2\gamma_3 h_1 H.$$

Applying (2) and the fact that $a_i \in C_G(h_j)$ and $b_i \in C_G(h_j)$ for every $1 \le i, j \le 3$ we can conclude A and B are really H-connected left transversals to H in G. $\square$

Claim 2. *For the above described group G the following statements are true:*

i) $\langle A, B\rangle = G$.

ii) $\mathrm{core}_G H = 1$.

iii) $G' \nleq N_G(H)$.

Proof. i) Since $b_i = a_i h_i$, $\beta_i = \gamma_i {h_i}^*$ and $H = \langle h_i, {h_i}^* \;/\; 1 \le i \le 3 \rangle$ it follows $\langle A, B \rangle = G$.

ii) It follows easily from the construction.

iii) We have $({h_2}^*)^{\gamma_1} = a_3 {h_2}^* h_3$, $({h_3}^*)^{\gamma_2} = a_1 {h_3}^*$, $({h_1}^*)^{\gamma_3} = a_2 {h_1}^* h_2$, ${a_1}^{\gamma_1} = a_1 z$, whence we can conclude $K \le G'H$. Since $({h_1}^*)^{a_1} = {h_1}^* z$ and $z \notin H$ we get $G' \not\le N_G(H)$. $\square$

Summarizing the properties of the obtained group G we get the following

Theorem 1.2.13 [Cs5, Statement 4.3]**.** *There exists a finite group G of order 2^{13} with the following properties: there is a subgroup H which is elementary abelian of order 2^6, there exist A and B H-connected transversals to H in G such that $\langle A, B \rangle = G$, $\mathrm{core}_G H = 1$ and $G' \not\le N_G(H)$. Furthermore the normal closure G_0 of H in G is of order 2^{10} and G/G_0 is elementary abelian of order 2^3.*

Proof. See Claims 1, 2 and the construction. $\square$

We get a negative answer to Problem 1.2.1:

Theorem 1.2.14 [Cs5, Statement 5.1]**.** *There exists a finite loop with abelian inner mapping group such that this loop is centrally nilpotent of class greater than two. More precisely, there is a loop Q of order 2^7 such that $\mathrm{Mlt}\, Q$ is of order 2^{13}, $\mathrm{Inn}\, Q$ is elementary abelian of order 2^6, for the normal closure M_0 of $\mathrm{Inn}\, Q$ in $\mathrm{Mlt}\, Q$, M_0 is of order 2^{10} and $\mathrm{Mlt}\, Q/M_0$ is elementary abelian of order 2^3, furthermore the nilpotency class of this loop Q is greater than two.*

Proof. The above described properties of our G imply that Theorem 1.1.1 is applicable for this G, consequently our G is isomorphic to the multiplication group $\mathrm{Mlt}\, Q$ of some loop Q of order 2^7. Since $H \cong \mathrm{Inn}\, Q$, $G_0 \cong M_0$ and $G' \not\le N_G(H)$, by using Theorem 1.2.13 and Corollary 1.2.3 we get our result. $\square$

Sufficient conditions for nilpotency class two

Using the before mentioned propositions concerning the minimal counterexample and nice subclasses we can extend our earlier results and obtain new sufficient conditions for nilpotency class two.

The special structure of abelian inner mapping group implies the nilpotency class two of the loop. The loop theoretical consequence of [NK2, Lemma 4.2] is that if $\mathrm{Inn}\, Q$ is elementary abelian of order p^2 for some prime p, then Q is centrally nilpotent of class at most two. The loop theoretical interpretation of [CsJK, Theorem 4.2] is the following: if $\mathrm{Inn}\, Q$ is the direct product of $C_{p_i} \times C_{p_i}$, where $1 \leq i \leq r$, $p_1, \ldots, p_r$ are different primes, then Q is centrally nilpotent of class at most two.

We extended this statement for a larger class of loops: If $\mathrm{Inn}\, Q$ is abelian and for every prime divisor p of the order of $\mathrm{Inn}\, Q$ the Sylow p-subgroup of $\mathrm{Inn}\, Q$ is isomorphic to a subgroup of $C_p \times C_p \times C_p$, then Q is centrally nilpotent of class at most two. We have found some properties of the normal closure M_0 of $\mathrm{Inn}\, Q$ in $\mathrm{Mlt}\, Q$ which guarantee the central nilpotency of class at most two of the finite loop Q. One of these properties is the cyclicity of $\mathrm{Mlt}\, Q/M_0$, the other is that $|\mathrm{Inn}\, Q|$ and $|\mathrm{Mlt}\, Q : M_0|$ are relatively prime.

We prove our results using H-connected transversals. Thus G and H are the usual group and abelian subgroup, A and B are H-connected transversals with $\langle A, B\rangle = G$. For the nilpotency class two we have to study the property $G' \leq N_G(H)$.

Theorem 1.2.15 [Cs4, Theorem 3.7]**.** *If $H \leq L_p \cong C_p \times C_p \times C_p$ or $H \leq M_p \cong C_{p^k} \times C_p$ where p is an arbitrary prime and $k \geq 2$, then $G' \leq N_G(H)$.*

Proof. Denote $\mathcal{F}_3{}^*$ the set of all $(G, H) \in \mathcal{F}$ which satisfy the conditions of our theorem. It is easy to see $\mathcal{F}_3{}^*$ is a nice subclass of $\mathcal{F}$.

Let G be a counterexample of smallest order. Clearly we can apply Proposition 1.2.11 for our G. Consequently $L_G(H) = \mathrm{core}_G H = 1$ (see Proposition 1.2.11 i)), hence Proposition 1.2.9 and Proposition 1.2.10 can be applied, too.

By Proposition 1.2.11 i) we have $L_G(H) = 1$. Let G_0 be the normal closure of H in G. Proposition 1.2.11 vi) and iv) give that G_0 is a p-subgroups and $G_0 \cap Z(G)$ is a nontrivial cyclic p-subgroup. $G_0/H(Z(G) \cap G_0)$ and $H/Z(G_0) \cap H$ are elementary abelian p-groups (see Proposition 1.2.11 viii), ix)).

Denote $H_1 = H \cap Z(G_0)$ and $Z_1 = Z(G) \cap G_0$ as in the proof of Proposition 1.2.11 xii). Then $Z(G_0) = Z_1 \times H_1$ (see Proposition 1.2.9 iii)).

We show $H_1 \neq 1$. Suppose $H_1 = Z(G_0) \cap H = 1$. Apply Proposition 1.2.10 for $Z^* = Z_1$, then $H^* = H \cap \mathrm{core}_G(Z^* H) = H_1 = 1$ (see Proposition 1.2.9 iv) and iii)). Using Proposition 1.2.10 iii) and v) we get $A \cap G_0 = B \cap G_0 \leq C_G(B) \cap C_G(A)$. Hence $G = \langle A, B\rangle$ implies $A \cap G_0 \leq Z(G)$. Since $G_0 = (A \cap G_0)H$ and $G' \leq G_0$ we can conclude $G' \leq N_G(H)$, a contradiction.

Using the structure of H we get H/H_1 is elementary abelian p-group of order at most p^2.

Case a) $|H/H_1| = p$.

As $Z(G_0) = Z_1 \times H_1$, Proposition 1.2.11 xi) gives $|G_0 : HZ_1| = |H : H_1| = p$. Using Proposition 1.2.11 xii) we can conclude $H^a \leq HZ_1$ for every $a \in A$. Hence $G_0 \leq HZ_1 \leq N_G(H)$.

Case b) $|H \,/\, H_1| = p^2$. Then clearly H_1 is a cyclic p-subgroup.

1) We claim $a^{-1}b \in H_1$ for every $a \in A$, $b \in B \cap aH$.

Assume there exist $a \in A$, $b \in B \cap aH$ such that $a^{-1}b \notin H_1$. Thus $h = a^{-1}b \in H \setminus H_1$. As in Proposition 1.2.11 v) and vii) denote Z_0 the subgroup of order p of $Z_1 = Z(G) \cap G_0$, $U = \mathrm{core}_G Z_0 H$ and $H_0 = U \cap H$. By Proposition 1.2.11 vii) $H_0 \leq H_1$ and since $\mathrm{core}_G H = 1$, then H_0 is of order p. By Proposition 1.2.11 v) we have $Z_0 H \trianglelefteq G_0$, Proposition 1.2.10 vii) gives $\mathrm{core}_{G_0} H = Z(G_0) \cap H = H_1$. We have $|G_0 : HZ_1| = |H : H_1| = p^2$ (see Proposition 1.2.11 xi)). By Proposition 1.2.11 vii) $G_0' \leq H_0 Z_0 \leq H_1 Z_1$. Thus $h^{a_j} \in hZ_0 H_1$ for every $a_j \in A \cap G_0$. Since Z_0 is of order p we can conclude easily that there exists $a_i \in (A \cap G_0) \setminus N_G(H)$ such that $h^{a_i} \in H$ and $H \cap H^{a_i} \leq H_1 \langle h \rangle$.

We claim $a_i = b_i$, where $b_i \in B \cap a_i H$.

Suppose $h_0 = {a_i}^{-1} b_i \neq e$. Applying Proposition 1.2.10 ii) for $Z^* = Z_0$, we get $h_0 \in \mathrm{core}_G Z_0 H$. Then Proposition 1.2.11 vii) gives $h_0 \in H_0$.

We show $a \in N_G(H_0)$. Using $[A, B] \leq H$ and $h^{a_i} \in H$ it follows ${h_0}^a = (a_i^{-1} b_i)^a = (a_i^{-1})^{bh^{-1}} {b_i}^a = h_i (a_i^{-1})^{h^{-1}} {b_i}^a \in H$, where $h_i \in H$. By Proposition 1.2.11 vii) $U = Z_0 H_0 \trianglelefteq G$, whence ${h_0}^a \in H_0$. Since $|H_0| = p$, then $a \in N_G(H_0)$.

Let $a_j \in A \cap G_0$, $b_j \in B \cap a_j H$. We show that $h^{a_j} \in H$.

Suppose $a_j = b_j$. Using $[A, B] \leq H$, it follows $h^{a_j} = (a^{-1}b)^{a_j} \in H$.

Suppose $a_j \neq b_j$. Applying Proposition 1.2.10 ii) to $Z^* = Z_0$, $H^* = (\mathrm{core}_G Z_0 H) \cap H = H_0$ (see Proposition 1.2.11 vii)) and we get $a_j = b_j h_j$ with $h_j \in H_0$. Using $[A, B] \leq H$, it follows $h^{a_j} = (a^{-1}b)^{a_j} \in (a^{-1})^{b_j h_j} bH$. As $a^{b_j} \in aH_0$ (see Proposition 1.2.10 iii), and $a \in N_G(H_0)$ we can conclude $h^{a_j} \in H$.

Thus $h^{a_j} \in H$ for every $a_j \in A \cap G_0$ consequently $h \in \mathrm{core}_{G_0} H$. By Proposition 1.2.10 vii) $\mathrm{core}_{G_0} H = Z(G_0) \cap H = H_1$, but $h \notin H_1$, a contradiction. Thus $a_i = b_i$.

Let $\alpha \in A$, $\beta \in B \cap \alpha H$. Since $[A, B] \leq H$ and $a_i = b_i$ it follows $(\alpha^{-1}\beta)^{a_i} \in H$. We have $H \cap H^{a_i} \leq H_1 \langle h \rangle$, consequently $\alpha^{-1}\beta \in H_1 \langle h \rangle$ for every $\alpha \in A$, $\beta \in B \cap \alpha H$.

Suppose $\alpha^{-1}\beta \in (H_1 \langle h \rangle) \setminus H_1$, then $\alpha = \beta w$ with $w \in (H_1 \langle h \rangle) \setminus H_1$, whence $h^\alpha = (a^{-1}b)^\alpha = (a^{-1})^{\beta w} b^\alpha$. Using $[A, B] \leq H$, $h^a \in H$ and $Z(G_0) = Z_1 \times H_1 \trianglelefteq G$ we get $h^\alpha \in HZ_1$.

Suppose $\alpha^{-1}\beta \in H_1$ i.e. $\alpha = \beta h^*$ with $h^* \in H_1$. Then $h^\alpha = (a^{-1}b)^\alpha = (a^{-1})^{\beta h^*} b^\alpha$. Since $h^* \in H_1$, $H_1 Z_1 \trianglelefteq G$ and $[A, B] \leq H$ we can conclude $h^\alpha \in HZ_1$.

Thus we get $h \in \mathrm{core}_G HZ_1$ and Proposition 1.2.9 iv) gives $h \in Z(G_0) \cap H = H_1$, a contradiction. Consequently $a^{-1}b \in H_1$ for every $a \in A$, $b \in B \cap aH$.

2) We show $A = B$. Assume there exist $\alpha_0 \in A$, $\beta_0 \in B \cap \alpha_0 H$ such that $\alpha_0 \neq \beta_0$. We have $\alpha_0^{-1}\beta_0 \in H_1$, by 1) H_1 is a cyclic subgroup of order p^ℓ and H_0 is the minimal subgroup of H_1, hence $[A, B] \leq H$ implies $\alpha_0 \in N_G(H_0)$. Let $\alpha_1 \in A$, $\beta_1 \in B \cap \alpha_1 H$ such that $\alpha_1 = \beta_1$. Using again $[A, B] \leq H$ it follows $(\alpha_0^{-1}\beta_0)^{\alpha_1} \in H$, since $\alpha_0^{-1}\beta_0 \in H_1$, then clearly $\alpha_1 \in N_G(H_0)$. Hence we get $A \subseteq N_G(H_0)$. As H is abelian we can conclude $H_0 \leq \mathrm{core}_G H = L_G(H) = 1$, a contradiction.

Thus $A = B$. $L_G(H) = 1$ implies that there exists $a_0 \in A$ such that $a_0 \notin N_G(H_0)$.

3) We show $a_0 a_j \in A$ for every $a_j \in A \cap G_0$.

Suppose $a^* \in A \cap a_0 a_j H$ and $a^* \neq a_0 a_j$ for some $a_j \in A \cap G_0$. Let $\gamma \in A$ be arbitrary, then by $[A, A] \leq H$ $(a_0 a_j)^\gamma = a_0 h a_j h_j$ where $h, h_j \in H$. Hence $d = (a_0 a_j)^{-1}(a_0 a_j)^\gamma = h^{a_j} h_j$. Since $HZ_0 \trianglelefteq G_0$ (see Proposition 1.2.11 v)) and $a_j \in A \cap G_0$ it follows $d \in HZ_0$. By using $(a^*)^\gamma \in a^* H$ we get $((a^*)^{-1} a_0 a_j)^\gamma \in HZ_0$ for every $\gamma \in A$, consequently $(a^*)^{-1} a_0 a_j \in H \cap \mathrm{core}_G HZ_0 = H_0$ (see Proposition 1.2.11 vii)). On the other hand, $((a^*)^{-1} a_0 a_j)^{a_0} \in H$ by $[A, A] \leq H$. Using $Z_0 H_0 \trianglelefteq G$ we get a contradiction with $a_0 \notin N_G(H_0)$.

4) We show ${a_0}^\gamma \in a_0 H_1$ for every $\gamma \in A$. As we have seen for $a_j \in A \cap G_0$ $(a_0 a_j)^\gamma = a_0 h a_j h_j$ with $h, h_j \in H$. Using $[A, A] \leq H$ and $a_0 a_j \in A$ it follows $h^{a_j} \in H$ for every $a_j \in A \cap G_0$, consequently $h \in Z(G_0) \cap H = H_1$ by Proposition 1.2.10 vii) and ${a_0}^\gamma \in a_0 H_1$ for every $\gamma \in A$.

5) We show $\gamma a_j \in A$ for every $\gamma \in A$, $a_j \in A \cap G_0$. Suppose there exist $\gamma \in A$, $a_j \in A \cap G_0$ such that $\gamma^* \in A \cap \gamma a_j H$ and $\gamma^* \neq \gamma a_j$. Applying Proposition 1.2.10 for $Z^* = Z(G) \cap G_0 = Z_1$, we get $H^* = (\mathrm{core}_G Z_1 H) \cap H = Z(G_0) \cap H = H_1$ (see Proposition 1.2.9 iv)), then Proposition 1.2.10 iii) and $A = B$ give $\gamma^{a_\ell} \in \gamma H_1$ for every $a_\ell \in A \cap G_0$. $Z(G_0) = H_1 \times Z_1$ implies $a_j \in C_G(H_1)$ whence $((\gamma^*)^{-1}\gamma a_j)^{a_\ell} \in H$. Consequently $(\gamma^*)^{-1}\gamma a_j \in \mathrm{core}_{G_0} H = H_1$ (see Proposition 1.2.10 vii)). On the other hand, using $\gamma^{a_0} \in \gamma H_1$ (see 4)) and $a_j \in C_G(H_1)$ it follows $((\gamma^*)^{-1}\gamma a_j)^{a_0} \in H$. Using $Z_0 H_0 \trianglelefteq G$ we get a contradiction with $a_0 \notin N_G(H_0)$.

6) We show $\gamma^{\gamma_1} \in \gamma H_1$ for arbitrary $\gamma, \gamma_1 \in A$. Let $a_j \in A \cap G_0$, then by $[A, A] \leq H$ we have $(\gamma a_j)^{\gamma_1} \in \gamma h_\gamma a_j H$ with $h_\gamma \in H$. Since $(\gamma a_j)^{\gamma_1} \in \gamma a_j H$ (see $\gamma a_j \in A$ by 5)) it follows $(h_\gamma)^{a_j} \in H$ for every $a_j \in A \cap G_0$, whence $h_\gamma \in \mathrm{core}_{G_0} H = H_1$ (see Proposition 1.2.10 vii)). As $\gamma^{\gamma_1} = \gamma h_\gamma$ we get $\gamma^{\gamma_1} \in \gamma H_1$ for any $\gamma, \gamma_1 \in A$.

We have $H_1 = H \cap Z(G_0)$, applying Proposition 1.2.9 vi) we can conclude $AZ(G_0) \trianglelefteq G$. Since $\langle A, B \rangle = G$ and $A = B$ it follows $AZ(G_0) = G$. Proposition 1.2.9 iii), v) imply $H = H_1$. Since $G' \leq G_0 \leq N_G(H_1)$ we get $G' \leq N_G(H)$. This is the final contradiction. □

Let H be a subgroup of a group G such that $G = \langle A, B \rangle$ for some H-connected transversals A and B in G. We say H satisfies condition (a), if $G' \leq N_G(H)$.

Theorem 1.2.16 [Cs4, Theorem 3.8]. *Let $H = H_1 \times \cdots \times H_r$ where $H_i \in \mathrm{Syl}_{p_i}(H)$.*

Assume for any $1 \le i \le r$ *every subgroup and every factor group of* H_i *satisfy* (a), *then* H *satisfies* $\langle a \rangle$ *too, i.e.* $G' \le N_G(H)$.

Proof. Denote $\mathcal{F}_4^*$ the set of all $(G,H) \in \mathcal{F}$ which satisfy the conditions of our theorem. It is easy to see $\mathcal{F}_4^*$ is a nice subclass of $\mathcal{F}$.

Let G be a counterexample of smallest order. Clearly $r \ge 2$ and G satisfies every condition of Proposition 1.2.11. Hence we get a contradiction to Proposition 1.2.11 v). □

Theorem 1.2.17 [Cs4, Theorem 3.9]**.** *Assume* $H = H_1 \times H_2 \times \cdots \times H_r$ *where* $H_i \in \mathrm{Syl}_{p_i}(H)$, $H_i \le L_i \cong C_{p_i} \times C_{p_i} \times C_{p_i}$ *or* $H_i \le M_i \cong C_{{p_i}^{k_i}} \times C_{p_i}$ *with* $k_i \ge 2$. *Then* $G' \le N_G(H)$.

Proof. It is clear by Theorem 1.2.15 and Theorem 1.2.16. □

During the study of minimal counterexample, the structure of the normal closure G_0 of H in G has played important role. I got some sufficient conditions for nilpotency class two concerning this normal closure G_0.

Theorem 1.2.18 [Cs4, Theorem 3.5]**.** *Assume* $|H|$ *and* $|G : G_0|$ *are relatively prime, where* G_0 *is the normal closure of* H *in* G. *Then* $G' \le N_G(H)$.

Proof. Denote ${\mathcal{F}_1}^*$ the set of all $(G,H) \in \mathcal{F}$ which satisfy the conditions of our theorem. It is easy to see that ${\mathcal{F}_1}^*$ is a nice subclass of $\mathcal{F}$.

Let G be a counterexample of smallest order. Then $G' \not\le N_G(H)$, and clearly we can apply Proposition 1.2.11 for our G. Consequently $L_G(H) = \mathrm{core}_G H = 1$ (see Proposition 1.2.11 i)), hence Proposition 1.2.9 and Proposition 1.2.10 can be applied, too.

We have G is solvable and G_0 is a normal p-subgroup of G (see Theorem 1.2.6 and Proposition 1.2.11 vi)). By the conditions of our theorem $G_0 \in \mathrm{Syl}_p(G)$. Applying Schur–Zassenhaus' theorem we get there exists a subgroup V such that $G_0V = G$ and $G_0 \cap V = 1$. Denote $M = (A \cap G_0)Z(G_0)$, let Z_0 and H_0 as in Proposition 1.2.11. As $H_0 \le Z(G_0)$ (see Proposition 1.2.11 vii)) using Proposition 1.2.10 v) for $Z^* = Z_0$ it follows M is an abelian normal p-subgroup of G.

We show $M = C_M(V)Z(G_0)$.

Let $a_i \in A \cap G_0$, $v \in V$, $b \in B \cap vH$. Then $v = bh$ for some $h \in H$. Using Proposition 1.2.10 iii) for $Z^* = Z_0$ we get ${a_i}^v = {a_i}^{bh} = (a_i h_0)^h$ with $h_0 \in H_0$. As $G_0' \le H_0 Z_0 \le Z(G_0)$ (see Proposition 1.2.11 vii)) it follows ${a_i}^h \in a_i Z(G_0)$, whence ${a_i}^v \in a_i Z(G_0)$. Consequently every element of V fixes $a_i Z(G_0)$ by the conjugation. Applying Glauberman's Theorem [Gla] we get there exists ${a_i}^* \in a_i Z(G_0)$ such that ${a_i}^* \in C_G(V)$. Since $M = (A \cap G_0)Z(G_0)$ and $a_i \in A \cap G_0$ is arbitrary we can conclude $M = C_M(V)Z(G_0)$.

As $G' \leq G_0$ and $V \cong G/G_0$ it follows V is abelian. We have $G_0/Z(G_0)$ is an elementary abelian p-group (see Proposition 1.2.11 x)). Clearly every element of V induces an automorphism on the elements of $G_0/Z(G_0)$ by the conjugation. We have $M = (A \cap G_0)Z(G_0)$ is an abelian normal p-subgroup of G. By the conditions $(|V|, p) = 1$, whence using Maschke's theorem for $G_0/Z(G_0)$ we get there exists a subgroup $\overline{W}$ of $G_0/Z(G_0)$ such that $\overline{W} = W/Z(G_0)$ and $WM = G_0$, $W \cap M = Z(G_0)$ and $V \leq N_G(W)$.

We have $G = G_0V$, $G_0 = WM$. Hence $G/M = WMV/M \cong MV/M \cap W$ holds, thus $G/M \cong MV/Z(G_0)$. Since $M = C_M(V)Z(G_0)$ and V is abelian it follows G/M is abelian, consequently $G' \leq M$. We have $[A, B] \leq H \cap G'$, whence $[A, B] \leq H \cap M$. $M = (A \cap G_0)Z(G_0)$ implies $[A, B] \leq H \cap Z(G_0)$. By using Proposition 1.2.9 vi) we can conclude $AZ(G_0) \trianglelefteq G$ and $BZ(G_0) \trianglelefteq G$. Clearly $(|AZ(G_0) : G_0 \cap AZ(G_0)|, |G_0 \cap AZ(G_0)|) = 1$ and $(|BZ(G_0) : G_0 \cap BZ(G_0)|, |G_0 \cap BZ(G_0)|) = 1$.

By Schur–Zassenhaus' theorem there exist subgroups W_1, W_2 of G such that $AZ(G_0) = W_1(G_0 \cap AZ(G_0))$, $W_1 \cap (G_0 \cap AZ(G_0)) = 1$, $BZ(G_0) = W_2(G_0 \cap BZ(G_0))$, $W_2 \cap (G_0 \cap BZ(G_0)) = 1$. These W_1 and W_2 are complements to G_0 in G, too. The Schur–Zassenhaus' theorem gives $W_2 = {W_1}^g$ for some $g \in G$. Since $AZ(G_0) \trianglelefteq G$ it follows $W_2 \leq AZ(G_0)$, consequently $AZ(G_0) = W_2(G_0 \cap AZ(G_0))$. We have $G_0 \cap AZ(G_0) = (A \cap G_0)(Z(G_0) \cap H)$, $G_0 \cap BZ(G_0) = (B \cap G_0)(Z(G_0) \cap H)$. Applying Proposition 1.2.10 for $Z^* = Z(G) \cap G_0$, then $\mathrm{core}_G(Z(G) \cap G_0)H = Z(G_0)$ (see Proposition 1.2.9 iv)), whence

$$H^* = H \cap (\mathrm{core}_G(Z(G) \cap G_0)H) = H \cap Z(G_0)$$

and Proposition 1.2.10 v) gives $(A \cap G_0)(Z(G_0) \cap H) = (B \cap G_0)(Z(G_0) \cap H)$. Consequently

$$AZ(G_0) = W_2(A \cap G_0)(Z(G_0) \cap H) = W_2(B \cap G_0)(Z(G_0) \cap H) = BZ(G_0).$$

Since $\langle A, B \rangle = G$, then $H = Z(G_0) \cap H$, which implies $G_0 \leq N_G(H)$. Using $G' \leq G_0$ (see Proposition 1.2.8 ii)) we can conclude $G' \leq N_G(H)$. This is the final contradiction. $\square$

Theorem 1.2.19 [Cs4, Theorem 3.6]**.** *Assume G/G_0 is cyclic, where G_0 is the normal closure of H in G. Then $G' \leq N_G(H)$.*

Proof. Denote ${\mathcal{F}_2}^*$ the set of all $(G, H) \in \mathcal{F}$ which satisfy the conditions of our theorem. It is easy to see that ${\mathcal{F}_2}^*$ is a nice subclass of $\mathcal{F}$.

Let G be a counterexample of smallest order. Clearly we can apply Proposition 1.2.11 for our G. Consequently $L_G(H) = \mathrm{core}_G H = 1$ (see Proposition 1.2.11 i)), hence Proposition 1.2.9 and Proposition 1.2.10 can be applied, too. Let $G = G_0 \langle c \rangle$. By Proposition 1.2.11 vii) we have ${G_0}' \leq Z(G_0) \leq N_{G_0}(H)$, whence $D = N_{G_0}(H)H^c \trianglelefteq G_0$. Let $d \in D$ be arbitrary, clearly $d = a_ih_i$ where $a_i \in A \cap D$, $h_i \in H$.

We have $c = bh$ for some $b \in B$, $h \in H$. Then $d^c = (a_ih_i)^{bh} = (a_ih^*)^h{h_i}^c$ where $h^* \in H^*$ (see Proposition 1.2.10 iii)). Since ${G_0}' \leq Z^*H^*$ (see Proposition 1.2.10 vi)) ${a_i}^h \in a_iZ^*H^*$ and $Z^*H^* \leq N_{G_0}(H)$ (see Proposition 1.2.10 i)) it follows $d^c \in D$, whence $c \in N_G(D)$. Using the definition of D we get $D \trianglelefteq G$. As $D \geq H$ and G_0 is the normal closure of H in G, we can conclude $D = G_0$. Hence using Lemma 1.2.4 it follows $G_0 = ((Z(G) \cap G_0)H)H^c$ contradicting to Proposition 1.2.11 xii). $\square$

Loop theoretical results

Our earlier theorems purely group theoretical results. Combining these theorems with Niemenmaa and Kepka's characterization theorem of multiplication groups of loops (see Theorem 1.1.1) and Corollary 1.2.3 we get the following loop theoretical results.

The loop theoretical interpretation of Theorem 1.2.15:

Corollary 1.2.20 [Cs4, Corollary 4.5]**.** *If Q is a finite loop and* $\operatorname{Inn} Q$ *is elementary abelian of order at most p^3, then Q is nilpotent of class at most two.*

Proof. See Theorem 1.1.1, Theorem 1.2.15 and Corollary 1.2.3. $\square$

The loop theoretical consequence of Theorem 1.2.17 is the following:

Corollary 1.2.21 [Cs4, Corollary 4.6]**.** *Let Q be a finite loop. Assume* $\operatorname{Inn} Q$ *is abelian and the Sylow subgroups of* $\operatorname{Inn} Q$ *are isomorphic to the subgroups of $C_p \times C_p \times C_p$. Then Q is centrally nilpotent of class at most two.*

Proof. See Theorem 1.1.1, Theorem 1.2.17 and Corollary 1.2.3. $\square$

The loop theoretical interpretation of Theorem 1.2.18 is the following:

Corollary 1.2.22 [Cs4, Corollary 4.3]**.** *Let Q be a finite loop with abelian inner mapping group* $\operatorname{Inn} Q$*. Denote M_0 the normal closure of* $\operatorname{Inn} Q$ *in* $\operatorname{Mlt} Q$*. Suppose* $|\operatorname{Inn} Q|$ *and* $|\operatorname{Mlt} Q : M_0|$ *are relatively prime. Then Q is centrally nilpotent of class at most two.*

Proof. See Theorems 1.1.1, 1.2.18, and Corollary 1.2.3. $\square$

The loop theoretical consequence of Theorem 1.2.19:

Corollary 1.2.23 [Cs4, Corollary 4.4]**.** *Let Q be a finite loop with abelian inner mapping group* $\operatorname{Inn} Q$*. Denote M_0 the normal closure of* $\operatorname{Inn} Q$ *in* $\operatorname{Mlt} Q$*. Suppose* $\operatorname{Mlt} Q/M_0$ *is cyclic. Then Q is centrally nilpotent of class at most two.*

Proof. See Theorem 1.1.1, Theorem 1.2.19 and Corollary 1.2.3. □

In our counterexample $\operatorname{Mlt} Q$ is a 2-group and $\operatorname{Mlt} Q/M_0$ is elementary abelian group of order 2^3. This fact proves that if in the latter two corollaries sufficient conditions concerning M_0 are not satisfied, then the nilpotency class 2 is not guaranteed.

1.3. Abelian groups as inner mapping groups

The structure of loops with abelian inner mapping groups became much more challenging after Kepka and Niemenmaa [NK1] published their paper on the nonexistence of loops with nontrivial cyclic inner mapping groups. They proved $\mathrm{Inn}\,(Q)$ is cyclic if and only if Q is an abelian group. The original proof was for finite loops, which was later changed to cover all loops [KN1]. Recently Drápal in [D2] offered a simplified proof. In [K1] Kepka proved that if $\mathrm{Inn}\,Q$ is abelian, then no nontrivial primary (p-) component of $\mathrm{Inn}\,Q$ is cyclic. After these results it is very natural to ask the following:

Problem: Which finite abelian groups can (or cannot) appear as the inner mapping group of some loop?

The question which finite abelian groups are possible as inner automorphism groups of groups was completely solved by Baer [Bae4]. The result is as follows:

Let G be a finite abelian group and let $G = C_1 \times \cdots \times C_n$ be the direct product of cyclic groups such that $|C_{i+1}|$ divides $|C_i|$ $(i = 1, \ldots, n-1)$. Then there exists a group H such that $\mathrm{Inn}\,H \cong G$ if and only if $n \geq 2$ and $|C_1| = |C_2|$.

The obtained results show that the situation in loop theory is similar as concerns the structure of finite abelian inner mapping groups of loops.

Even the conjecture – as M. Niemenmaa formulated it in [N1] – is the following:

Conjecture. If Q is a loop and $\mathrm{Inn}\,Q = C_1 \times C_2 \times \cdots \times C_h$ is a direct product of cyclic subgroups such that $|C_{i+1}|$ divides $|C_i|$ $(i = 1, \ldots, n-1)$, then $n \geq 2$ and $|C_1| = |C_2|$.

The direction of research is mainly determined by the above mentioned Baer theorem and conjecture.

In [N3] M. Niemenmaa showed that for a finite loop Q, $\mathrm{Inn}\,Q$ cannot be isomorphic to $C_n \times D$, where C_n is a cyclic group of order n and D is a finite abelian group such that $|D|$ is relatively prime to n. Later Kepka got this result for infinite loops, too. In [N4] Niemenmaa proved that if Q is a finite loop, then $\mathrm{Inn}\,Q$ is never isomorphic to the direct product $C_{p^k} \times C_p$, where p is an odd prime and $k \geq 2$. By [CsJK, Theorem 4.1] it is true for every prime p. In the same paper [N4] Niemenmaa showed that if Q is a finite loop, p and q are different primes such that p is odd and q does not divide $|Q|$, then $\mathrm{Inn}\,Q$ is not isomorphic to $(C_{p^k} \times C_p) \times D$ where $k \geq 2$ and D is an abelian q-subgroup. In [CsK] we gave a generalization of this result. Using [CsK, Remark 5.3, Remark 5.5] it is easy to see that if Q is a finite loop and p is a prime such that p does not divide $|Q|$, then $\mathrm{Inn}\,Q$ cannot be isomorphic to an abelian group which contains a p-subgroup. The loop theoretical consequence of [CsJK, Theorem 4.2]: If Q is a finite loop, then $\mathrm{Inn}\,Q \ncong C_{p_1^{m_1}} \times C_{p_1} \times C_{p_2^{m_2}} \times C_{p_2} \times \cdots \times C_{p_r^{m_r}} \times C_{p_r}$ where $r \geq 1$, $p_1, \ldots, p_r$ are different primes $m_1 \geq 2$, $m_2 \geq 0, \ldots, m_r \geq 0$.

We extend these earlier results. Our problem is closely related to the question of nilpotency.

Our problem is closely related to the question of nilpotency class. Our proofs heavily depend on results obtained in connection with nilpotency class two.

We use again the theory of connected transversals. Thus G is a finite group with abelian subgroup H. There exist A and B H-connected transversals such that $\langle A, B\rangle = G$. Then H satisfies condition (b) means that $L_G(H) \neq 1$, consequently H is not isomorphic to the inner mapping group of some loop by Theorem 2.1.1.

Theorem 1.3.1 [Cs4, Theorem 3.10]**.** *Assume $G' \leq N_G(H)$ and there exists $P_1 \in \mathrm{Syl}_{p_1}(H)$ such that P_1 satisfies* (b)*. Then H satisfies* (b)*, too, i.e. $L_G(H) \neq 1$.*

Proof. Let G be a counterexample of smallest order. Thus $L_G(H) = \mathrm{core}_G H = 1$ and clearly $H \neq P_1$. Let $P_2 \in \mathrm{Syl}_{p_2}(H)$, such that $p_2 \neq p_1$. Since $G_0 = G'H$ (see Proposition 1.2.8 ii)) and $G' \leq N_G(H)$ it follows $G_0 \leq N_G(H)$. If $P_2 \in \mathrm{Syl}_{p_2}(G_0)$, then $N_G(H) = H \times Z(G)$ (see Lemma 1.2.4) implies P_2 is a characteristic subgroup of G_0. As $G_0 \trianglelefteq G$ we can conclude $P_2 \trianglelefteq G$ contradicting to $\mathrm{core}_G H = 1$. Hence there exists $P_2^* \in \mathrm{Syl}_{p_2}(G_0)$ such that P_2^* contains P_2 but $P_2^* \neq P_2$. We have $\mathrm{core}_G H = 1$, whence Lemma 1.2.4 gives $G_0 = (Z(G) \cap G_0) \times H$. Let $Z_2 \in \mathrm{Syl}_{p_2}(Z(G) \cap G_0)$ (obviously $Z_2 \neq 1$) and $U_2 = \mathrm{core}_G Z_2 H$. Clearly $U_2 \cap P_1 \trianglelefteq G$, whence by $\mathrm{core}_G H = 1$, it follows $U_2 \cap P_1 = 1$. Denote f the natural homomorphism of G onto G/U_2. Then $f(A)$ and $f(B)$ are $f(H)$-connected transversals in G/U_2 by Lemma 1.2.7. Since $HU_2/U_2 \geq P_1U_2/U_2 \cong P_1$ it is easy to see that G/U_2 satisfies every condition of our theorem. The minimality of G implies $\mathrm{core}_{G/U_2} HU_2/U_2 \neq 1$, contradicting the choice of U_2. $\square$

Theorem 1.3.2 [Cs4, Theorem 3.11]**.** *Assume $H = H_1 \times H_2 \times \cdots \times H_r$ where $H_i \in \mathrm{Syl}_{p_i}(H)$, $H_1 \cong C_{{p_1}^{k_1}} \times C_{p_1}$ with $k_1 \geq 2$, and for every $2 \leq i \leq r$ $H_i \leq L_i \cong C_{p_i} \times C_{p_i} \times C_{p_i}$ or $H_i \leq M_i \cong C_{{p_i}^{k_i}} \times C_{p_i}$ with $k_i \geq 2$. Then H satisfies $L_G(H) \neq 1$.*

Proof. Our G satisfies the conditions of Theorem 1.2.17, hence we can conclude $G' \leq N_G(H)$. By Theorem 1.2.15 H_1 satisfies (b), hence applying Theorem 1.3.1 we get our statement. $\square$

Loop theoretical result

Theorem 1.3.1 is purely group theoretical. results. Combining this with Niemenmaa and Kepka's characterization theorem of multiplication groups of loops (see Theorem 1.1.1) we get the following:

Corollary 1.3.3 [Cs4, Corollary 4.1]**.** *If Q is a finite loop, then $\mathrm{Inn}\, Q \ncong H_1 \times H_2 \times \cdots \times H_r$ where $H_1 \cong C_{{p_1}^{k_1}} \times C_{p_1}$ with $k_1 \geq 2$ and for every $2 \leq i \leq r$ $H_i \leq L_i \cong$*

$C_{p_i} \times C_{p_i} \times C_{p_i}$ *or* $H_i \leq M_i \cong C_{{p_i}^{k_i}} \times C_{p_i}$ *with* $k_i \geq 2$ *and* $p_1, p_2, \ldots, p_r$ *are different primes.*

Recently M. Niemenmaa in [N8] extended this result. He proved that the inner mapping group cannot be isomorphic to the direct product $C_{p^k} \times C_p \times C_p$ where p is an odd prime and $k \geq 2$. His proof heavily depends on my result.

1.4. LCC loops and nilpotency class

A loop Q is said to be conjugacy closed (C C) if the set $\{L_x \mid x \in Q\}$ and $\{R_x \mid x \in Q\}$ are closed under the conjugation. The concept of conjugacy closedness was formulated first by Soikis [So] and later independently by Goodaire and Robinson [GR]. I have to mention the papers of Kinyon, Kunen and Phillips [KKP], P. Nagy and Strambach [NStr] whose results are relevant in this area.

A loop Q is called left conjugacy closed (LCC), if the set $\{L_x \mid x \in Q\}$ is closed under the conjugation, i.e. for all $a, b \in Q$ there exists $c \in Q$ such that $L_a L_b L_a^{-1} = L_c$.

LCC loops were also introduced by Soikis [So]. Basarab's paper [Bas], then Drápal's paper [D3], P. Nagy and K. Strambach's paper [NStr] can be considered main sources of results in connection with LCC loops. The large part of this latter paper is concerned with geometry of LCC loops. As a Bol loop Q is LCC if $x^2 \in N_\lambda$ for all $x \in Q$, we have to mention the paper of G. Nagy and his joint paper with H. Kiechle [KiN].

A. Drápal who studied in [D1] the relationship within multiplication groups of conjugacy closed loops, in his other paper [D3] could transfer some basic facts from CC loops to LCC loops and he studied the properties of LCC loops from really algebraic aspects.

In case of LCC loops with A. Drápal we analysed the converse of Bruck's result:

Problem 1.4.1. *Whether every LCC loop with abelian inner mapping groups has nilpotency class at most two?*

We obtained positive answer to this problem. Originally it was solved by group theoretical methods by using H-connected transversals.

I present here this group theoretical proof which is different from the proof presented in our paper. In fact, the idea of both proofs is the same. By comparing them one can see how the language of H-connected transversals differs from the language (more or less) of pure loop theory. I hope that the proof to follow will be of help to a reader that is mainly working in group theory.

For this proof we need one of the basic results of Drápal concerning LCC loops:

Theorem 1.4.1 [D3, Theorem 2.8]**.** *If Q is an LCC loop, then there exists a unique homomorphism $\Lambda : \mathcal{L} \to \operatorname{Inn} Q$ ($\mathcal{L}$ is the left multiplication group of Q) that maps L_x to T_x for each $x \in Q$. This homomorphism is the identity on $\mathcal{L}_1 = \mathcal{L} \cap \operatorname{Inn} Q$ and* $\operatorname{Ker} \Lambda = \{R_x / x \in Q\} \cap \mathcal{L} = Z(\mathcal{L})$*, furthermore if $R_x \in Z(L)$, then $x \in N_\varrho$.*

(In [Cs6] I studied different possible extensions of this homomorphism Λ and the uniqueness of these extensions.)

The converse of Bruck's result in case of LCC loops was originally solved by group theoretical methods using H-connected transversals. Thus we proved

Theorem 1.4.3. *Let G be a finite group with abelian subgroup H and $\mathrm{core}_G H = 1$. There exist A and B H-connected transversals ($[A,B] \leq H$) such that $\langle A, B\rangle = G$. Assume $A^a = A$ for every $a \in A$. Then $G' \leq N_G(H)$.*

Proof of Theorem 1.4.3. Let G be a counterexample of minimal order, thus $G' \not\leq N_G(H)$. By Theorem 1.2.6 G is solvable. Lemma 1.2.5 gives H is subnormal in G. We have $N_G(H) = H \times Z(G)$ and $Z(G) \subseteq A \cap B$ (see Lemma 1.2.4), consequently $Z(G) \neq 1$.

Denote G_0 the normal closure of H in G.

Claim 1. i) *If $H \leq M \trianglelefteq G$, then G/M is abelian.*

ii) *$G_0 = G'H$.*

iii) *$G_0 \neq G$.*

Proof. See Proposition 1.2.8. $\square$

Claim 2. i) *$A \cap H = B \cap H = 1$.*

ii) *$Z(G_0) = (Z(G) \cap G_0) \times (Z(G_0) \cap H)$, $Z(G) \cap G_0 \neq 1$.*

iii) *$\mathrm{core}_G(Z(G) \cap G_0)H = Z(G_0) \neq 1$.*

Proof. See Proposition 1.2.9 i), iii), iv). $\square$

Denote $Z_0 = Z(G) \cap G_0$ and $H_0 = Z(G_0) \cap H$. Thus $\mathrm{core}_G(Z(G) \cap G_0)H = Z(G_0) = Z_0 \times H_0$.

Claim 3. *$HZ_0 \trianglelefteq G_0$.*

Proof. Denote f the natural homomorphism of G onto $G/Z(G_0)$. Then $f(A)$ and $f(B)$ are $f(H)$-connected transversals in $G/Z(G_0)$ (see Lemma 1.2.7). Clearly $G/Z(G_0)$ satisfies every condition of our theorem. The minimality of G implies the commutator subgroup $(G/Z(G_0))' \leq N_{G/Z(G_0)}(HZ(G_0)/Z(G_0))$ whence $HZ(G_0)/Z(G_0) \trianglelefteq G'H/Z(G_0)$. As $Z(G_0)H = Z_0H$ and $G_0 = G'H$ (see Claim 1. ii)) we get our statement. $\square$

Claim 4. i) *If $a \in A \cap G_0$, $b \in B \cap aH$, then $a^{-1}b \in H_0$.*

ii) *$[A \cap G_0, B] \leq H_0$, $[B \cap G_0, A] \leq H_0$.*

Proof. Claim 3 and Proposition 1.2.10 ii), iii) give our statement. $\square$

By using Niemenmaa's and Kepka's characterization theorem we have G is the multiplication group of some left conjugacy closed loop Q, H is the inner mapping group. $A = \{L_x \mid x \in Q\}$, $B = \{R_x \mid x \in Q\}$. Denote $L = \langle A\rangle$, using homomorphism Λ (in Theorem 1.4.2) we have $\operatorname{Ker}\Lambda = Z(L) = L\cap B$. Put $Z^*(L) = Z(L)\cap G_0$.

Claim 5. *$L_1 \le H_0$, where $L_1 = L \cap H$.*

Proof. First we show $Z^*(L)H_0 \trianglelefteq G$. We have $Z(G)\cap G_0 = Z_0 \le Z^*(L)$, $Z_0H_0 \trianglelefteq G$, whence $Z^*(L)H_0 \le G$. Let $b \in B \cap Z^*(L)$ and $a \in A \cap bH$. By Claim 4 i) $b = ah_0$ with $h_0 \in H_0$. Let $b_1 \in B$, then $b^{b_1} = (ah_0)^{b_1} = ah_1h_0^{b_1}$ where $h_1 \in H_0$ by Claim 4 ii). Since $Z_0H_0 \trianglelefteq G$ we can deduce $b^{b_1} \in bH_0Z_0$. As $b \in Z^*(L)$ and $\langle A, B\rangle = G$, it follows easily $Z^*(L)H_0 \trianglelefteq G$. Thus $Z^*(L)H \le G$.

Now we show $Z^*(L)H = G_0$. Let $a_0 \in A$, $a_1 \in A\cap b_1H$. Using $L/Z(L) \cong \operatorname{Inn}\Lambda \le H$ and the commutativity of H, $G' \le G_0$ implies $a_1^{-1}a_1^{a_0} \in G_0 \cap Z(L) = Z^*(L)$. By $[A, B] \le H$ $b_1^{-1}b_1^{a_0} \in H$ is true, consequently $(a_1^{-1}b_1)^{a_0} \in Z^*(L)H$. $G = AH$ implies $(a_1^{-1}b_1)^g \in Z^*(L)H$ for every $g \in G$, whence $a_1^{-1}b_1 \in \operatorname{core}_G Z^*(L)H$ for every $a_1 \in A$, $b_1 \in B \cap A_1H$. Denote $F = \operatorname{core}_G Z^*(L)H$. Since $FL \le G$ and $A \subseteq FL$ using $\langle A, B\rangle = G$ we deduce $FL = G$. As $Z^*(L) \trianglelefteq L$ and $F \le Z^*(L)H$ we can see $Z^*(L)H \trianglelefteq G$. Hence $Z^*(L)H \ge G_0$ (see Prop. 1.2.8). Using $Z^*(L) = Z(L) \cap G_0$ it follows $Z^*(L)H = G_0$. $Z(L) = L \cap B$ implies $Z^*(L) = B \cap G_0$. Since $G = LH$ and H is abelian, then $L \cap HL_1 \le Z(G_0) \cap H = H_0$. $\square$

Claim 6. i) *$[A, B] \le H_0$.*

ii) *$LH_0 \le G$.*

Proof. i) Let $a_1, a_2 \in A$. Then $a_1a_2 = a_3h$ with $a_3 \in A$, $h \in H \cap L = L_1 \le H_0$ (see Claim 5). Let $b \in B$ be arbitrary. Using $[A, B] \le H$ we get ${a_1}^b = a_1h_1$, ${a_2}^b = a_2h_1$, ${a_3}^b = a_3h_3$ with $h_1, h_2, h_3 \in H$. Hence $(a_1a_2)^b = a_1a_2h_1^{a_2}h_2 = a_3h_3h^b$. Since $Z(G_0) = Z_0 \times H_0 \trianglelefteq G$, then $h^b \in H_0Z_0$, which implies $h_1^{a_2} \in HZ_0$ for every $a_2 \in A$. As $G = AH = HA$ it follows $h_1 \in \operatorname{core}_G HZ_0$, $\operatorname{core}_G HZ_0 = H_0Z_0$ (see Claim 2 iii)). Thus ${a_1}^b \in a_1H_0$ for every $a_1 \in A$, $b \in B$ as required.

ii) We have $Z(G_0) = Z_0H_0 \trianglelefteq G$, $Z_0 \le Z(G) \subseteq A \cap B \subseteq L$, hence $LH_0 = LZ_0H_0 \le G$. $\langle A, B\rangle = G$ and $[A, B] \le H_0$ (see i)) imply our statement. $\square$

Claim 7. i) *Denote $H_1 = \langle a_0^{-1}b_0 \mid a_0 \in A,\ b_0 \in Ba_0H\rangle$. Then $H_0H_1 = H$.*

ii) *$[A, A] \le Z(G)$.*

Proof. i) Since $LH_0 \le G$ then $L_0 = LH_0H_1 \le G$. By the definition of H_1, $B \subseteq L_0$ and using $\langle A, B\rangle = G$ we can deduce $L_0 = G$. As we have $L_1 \le H_0$ (see Claim 5) consequently $H_0H_1 = H$.

ii) Let $a, a_1 \in A$. Using our homomorphism Λ and the commutativity of H we have $a^{-1}a^{a_1} \in \operatorname{Ker}\Lambda \cap G_0 = L \cap B \cap G_0 = Z^*(L)$, thus $a^{-1}a^{a_1} = b^* \in B$. First we show $b^* \in N_G(H)$. Since $a^{a_1} \in A$ it follows $ab^* \in A$. Using $[A, B] \le H_0$ (see Claim 6 i)) ${b_0}^a \in b_0H$ and $b_0^{ab^*} \in b_0H$ for every $b_0 \in B$. Let $a_0 \in A \cap b_0H$, then $a_0^{b^*} \in a_0H_0$ by Claim 6 i) consequently $(a_0^{-1}{b_0}^a)^{b^*} \in H$. $H_0 \le Z(G_0)$ implies $b^* \in C_G(H_0)$, hence we get $(b_0^{-1}{b_0}^a)^{b^*} = b_0^{-1}{b_0}^a$ and we can conclude $(a_0^{-1}b_0)^{b^*} \in H$ for every $a_0 \in A$, $b_0 \in B \cap a_0 H$. By i) we have $H = H_0 H_1$, since $b^* \in C_G(H_0)$ and $H_1^{b^*} \le H$, $b^* \in N_G(H)$ holds. As $N_G(H) = H \times Z(G)$ and $Z(G) \subseteq A \cap B$ (see Lemma 1.2.4) it follows $b^* = a^{-1}a^{a_1} \in Z(G)$ for every $a, a_1 \in A$. □

Claim 8. *$H^g \le N_G(H)$ for every $g \in G$.*

Proof. Let $\alpha_0 \in A$, $\beta_0 \in B \cap \alpha_0 H$, $g \in G$. As $G = AH = HA$ we have $g = w\alpha$ with $w \in H$, $\alpha \in A$. Clearly $(\alpha_0^{-1}\beta_0)^g = (\alpha_0^{-1}\beta_0)^\alpha$. By using $[A, B] \le H_0$ and $[A, A] \le Z(G)$ (see Claim 6 i), 7 ii)) we can conclude $(\alpha_0^{-1}\beta_0)^g \in H \times Z(G)$. We have $H \times Z(G) = N_G(H)$ (Lemma 1.2.4). Since $Z(G_0) = H_0 Z_0 \trianglelefteq G$ and $H_0 H_1 = H$ (see Claim 2, 7 i)) it is easy to see $H^g \le N_G(H)$. □

Thus the normal closure G_0 of H in G is in $N_G(H)$. Hence Claim 1 implies $G' \le N_G(H)$. This is the final contradiction. □

Using Theorem 1.1.1 we get

Theorem 1.4.4. *Let Q be an LCC loop with abelian inner mapping group. Then the nilpotency class of Q is at most two.*

In case of LCC loops in our paper we proved the original Bruck's statement, so we got the following equivalency:

Theorem 1.4.5 [CsD1, Theorem 3.7]**.** *Let Q be a left conjugacy closed loop. Then $Q/Z(Q)$ is an abelian group (i.e. $\operatorname{cl} Q \le 2$) if and only if $\operatorname{Inn} Q$ is abelian.*

Our next problem was: which properties of a nilpotent loop of class two guarantee the left conjugacy closedness.

We obtained the following:

Theorem 1.4.6 [CsD1, Corollaries 3.2, 3.4, Theorem 4.4]**.** *If Q is a loop of nilpotency class two then the following statements are equivalent:*

i) *Q is left conjugacy closed.*

ii) *$\mathcal{L}/Z(\operatorname{Mlt} Q)$ is abelian.*

iii) *For all $x, y \in Q$ $\quad L(x,y) = L(y,x) \quad (L(x,y) = {L_{xy}}^{-1} L_x L_y)$.*

iv) *$[x, y, z] = [x, z, y]$ for every $x, y, z \in Q$.*

v) *$[x, y, z] = [xy, z]^{-1}[x, z]^{-1}[y, z]^{-1}$ for all $x, y, z \in Q$.*

After introducing a general construction for LCC loops, [CsD1] contains the description of the structure of every nilpotent LCC loop of order p^2.

Later in [CsD2] with A. Drápal we studied the structural properties of those LCC loops whose left multiplication group is normal in the whole multiplication group.

1.5. Buchsteiner loops and loops that are abelian groups over the nucleus

In this chapter we present a class of loops satisfying a certain identity which was introduced by H. H. Buchsteiner in 1976. Though his paper left many problems open, along time this variety of loops has not received much attention. Recently we investigated fundamental properties of these Buchsteiner loops, which prove they are closely connected to conjugacy closed loops. One of these main results is that the Buchsteiner loops over the nucleus are abelian groups as the conjugacy closed loops, even we could prove that the exponent of this factor is 4. A. Drápal and M. Kinyon could construct an example [CsDK] where the factor achieves this exponent. Buchsteiner did not work with the notion of conjugacy closedness and it turned out that the examples he constructed in [Buch] are all conjugacy closed. With A. Drápal we presented the first example [CsD4] of a proper (non-conjugacy closed) Buchsteiner loop. At the same time with A. Drápal we proved [CsD3] that a Buchsteiner loop over the center is conjugacy closed, and showed that a loop Q has to be conjugacy closed modulo $Z(Q)$ whenever all mappings $L(x, y)$ and $R(x, y)$ are automorphisms of Q, $N(Q) \trianglelefteq Q$, $Q\,/\,N(Q)$ is an abelian group and left multiplication groups is a normal subgroup in the multiplication group. Here we give some sufficient and in some cases necessary conditions for the conjugacy closedness of $Q/Z(Q)$ provided that Q/N is an abelian group. We show that if for some loop Q, Q/N and Inn Q are abelian groups, then $Q/Z(Q)$ is a CC loop, consequently Q has nilpotency class at most three. We give additionally some reasonable conditions which imply the nilpotency of the multiplication group of class at most three. We describe the structure of Buchsteiner loops with abelian inner mapping groups.

Basic definitions and results

We present some earlier mentioned and some new definitions and propositions that we need later.

Definition. We say that Q is an A_l-loop (A_r-loop) if $\mathcal{L}_1 \leq \mathrm{Aut}\ Q$ ($\mathcal{R}_1 \leq \mathrm{Aut}\ Q$). A loop Q is an $A_{r,l}$-loop if it is both an A_r-loop and an A_l-loop. ($\mathcal{L}_1 = \mathcal{L} \cap \mathrm{Aut}\, Q$, $\mathcal{R}_1 = \mathcal{R} \cap \mathrm{Aut}\, Q$.)

Proposition 1.5.1 [Cs7, Proposition 2.3]**.** *Let Q be a loop. Then*

i) $C_{\mathrm{Mlt}\ Q}(\mathcal{R}) = \{L_c \mid c \in N_\lambda\}$,

$C_{\mathrm{Mlt}\ Q}(\mathcal{L}) = \{R_d \mid d \in N_\varrho\}$.

ii) *If* $\mathcal{R} \trianglelefteq \mathrm{Mlt}\ Q$*, then* $C_{\mathrm{Mlt}\ Q}(\mathcal{R}) \trianglelefteq \mathrm{Mlt}\ Q$ *and* $N_\lambda \trianglelefteq Q$*.*

iii) *If* $\mathcal{L} \trianglelefteq \mathrm{Mlt}\, Q$*, then* $C_{\mathrm{Mlt}\ Q}(\mathcal{L}) \trianglelefteq \mathrm{Mlt}\ Q$ *and* $N_\varrho \trianglelefteq Q$*.*

iv) $A^*A = A$*,* $B^*B = B$*, where* $A^* = C_{\mathrm{Mlt}\ Q}(\mathcal{R})$*,* $B^* = C_{\mathrm{Mlt}\ Q}(\mathcal{L})$*.*

Proof. i), ii), iii): see [CsDK, Lemma 1.7]. iv) is trivial. □

Buchsteiner loops and loops that are abelian groups modulo the nucleus

Buchsteiner loops are defined by the identity

$$x \setminus (xy \cdot z) = (y \cdot zx)/x. \tag{B}$$

Here $a \setminus b$ denotes the unique solution x to $ax = b$, while b/a denotes the unique solution y to $ya = b$. We call (B) the Buchsteiner law since Hans-Hennig Buchsteiner initiated their study in [Buch].

Rewriting the Buchsteiner law (B) in terms of translations immediately yields

Lemma 1.5.2 [Cs7, Lemma 3.1]**.** *Q is a Buchsteiner loop, the Buchsteiner law is equivalent to each of the following:*

$$L_x{}^{-1} R_z L_x = R_x{}^{-1} R_{zx} \quad \textit{for all } x, z \in Q,$$
$$R_x{}^{-1} L_y R_x = L_x{}^{-1} L_{xy} \quad \textit{for all } x, y \in Q.$$

Proposition 1.5.3 [Cs7, Proposition 3.2]**.** *Let Q be a Buchsteiner loop. Then the following statements are true:*

i) $\mathcal{L} = \langle A \rangle \trianglelefteq \mathrm{Mlt}\ Q$*,*
$\mathcal{R} = \langle B \rangle \trianglelefteq \mathrm{Mlt}\ Q$*,*
$[A, B] = \mathcal{R}_1 = \mathcal{L}_1$*.*

ii) *The nucleus* $N \trianglelefteq Q$ *and*

$$N = N_\lambda = N_\mu = N_\varrho.$$

Put $A_0 = \{L_c \mid c \in N\}$*,* $B_0 = \{R_d \mid d \in N\}$*. Then*

$$A_0 = C_{\mathrm{Mlt}\ Q}(\mathcal{R}), \quad A_0 \trianglelefteq \mathrm{Mlt}\ Q,$$
$$B_0 = C_{\mathrm{Mlt}\ Q}(\mathcal{L}), \quad B_0 \trianglelefteq \mathrm{Mlt}\ Q.$$

iii) Q/N *is an abelian group of exponent four (an example in which this exponent is achieved is constructed in* [CsDK]*).*

iv) Q is an $A_{r,l}$-loop.

v) $Q/Z(Q)$ is a CC loop.

Proof. i) See [CsDK, Corollary 1.3].

ii) See [CsDK, Corollaries 1.6, 1.8].

iii) See [CsDK, Theorem 7.14].

iv) See [CsDK, Corollary 5.4].

v) See [CsD3, Theorem 3.5]. □

Buchsteiner loops are abelian groups modulo the nucleus. We shall now state some basic properties of loops that are abelian groups over the nucleus.

Lemma 1.5.4 [Cs7, Lemma 3.3]**.** *Let Q be a loop such that $N \trianglelefteq Q$, Q/N is an abelian group. Set $A_0 = \{L_c \mid c \in N\}$, $B_0 = \{R_d \mid d \in N\}$. Let $G = \mathrm{Mlt}\, Q$ and $H = \mathrm{Inn}\, Q$. Then the following statements are true:*

i) $\mathrm{core}_G A_0 H \supseteq [A, B] \cup (\langle A\rangle \cap H) \cup (\langle B\rangle \cap H)$.

ii) *Put $G_1 = A_0H = B_0H$.*

Then $G_1 \trianglelefteq G$ and G/G_1 is abelian.

iii) $Z(G_1) = Z(G) \times (Z(G_1) \cap H)$.

iv) $A_0 \trianglelefteq G$, $B_0 \trianglelefteq G$.

v) $A_0B_0 \le C_G([A, B])$.

vi) *Suppose $h \in H \cap \mathrm{Aut}\, Q$, $a \in A$, $b \in B$. Then $h^a = h\alpha_0$, $h^b = h\beta_0$ with $\alpha_0 \in A_0$, $\beta_0 \in B_0$.*

vii) *If $h \in H \cap \mathrm{Aut}\, Q$, then $h \in C_G([A, B])$.*

Proof. i) By $N \trianglelefteq Q$, we have $A_0H \le G$, $B_0H \le G$. Using Q/N is abelian it follows $\mathrm{core}_G A_0H \supseteq [A, B] \cup (\langle A\rangle \cap H) \cup (\langle B\rangle \cap H)$.

ii) By i) clearly $\langle A\rangle \cap A_0H \trianglelefteq \langle A\rangle$. Let $a \in A$, $b \in B \cap aH$. Then using $[A, B] \le H$ we get $(a^{-1}b)^{a^*} \in A_0H$ for every $a^* \in A$, in similar way $(a^{-1}b)^{b^*} \in B_0H (= A_0H)$ for every $b^* \in B$. Since $G = \langle A, B\rangle$ and $H = \left\langle a^{-1}b,\ \langle A\rangle \cap H,\ \langle B\rangle \cap H \mid a \in A,\ b \in B \cap aH \right\rangle$ by Proposition 1.1.3 we can conclude that $G_1 \trianglelefteq G$.

iii) Using $Z(G_1) \le N_G(H)$ and $N_G(H) = Z(G) \times H$ (see [NK1, Proposition 2.7]) it follows easily.

iv) By [CsD3, Lemma 1.7] and by i) $A_0 \trianglelefteq \langle A\rangle$. Since $A_0 \le C_G(B)$ (see Proposition 1.5.1) and $\langle A, B\rangle = G$ it follows $A_0 \trianglelefteq G$. In a similar way $B_0 \trianglelefteq G$ holds.

v) Using $A_0 \trianglelefteq G$ and $A_0 \le C_G(B)$ we can see easily $A_0 \le C_G([A, B])$, and similarly $B_0 \le C_G([A, B])$.

vi) By [CsD3, Lemma 1.2] $a^h \in A$, $b^h \in B$. Since $G_1 \trianglelefteq G$, $A_0 \trianglelefteq G$, $B_0 \trianglelefteq G$, and $A_0 A = A$, $B_0 B = B$ we get our statement.

vii) Using vi) and $A_0 \leq C_G(\langle B \rangle)$, $B_0 \leq C_G(\langle A \rangle)$ it follows easily. □

The conjugacy closed loops (CC loops) Q satisfy the following properties:

$$\langle A \rangle \trianglelefteq \mathrm{Mlt}\ Q, \quad \langle B \rangle \trianglelefteq \mathrm{Mlt}\ Q,$$
$$Q \text{ is an } A_{r,l}\text{-loop, furthermore } N \trianglelefteq Q,\ Q/N \text{ is an abelian group.}$$

In [CsD3] we studied the converse of this result, i.e. those loops satisfying these conditions and we got that they are very close to the CC loops:

Proposition 1.5.5 [CsD3, Theorem 3.1]**.** *Let Q be a loop such that $N \trianglelefteq Q$, $\langle A \rangle \trianglelefteq \mathrm{Mlt}\ Q$, $\langle B \rangle \trianglelefteq \mathrm{Mlt}\ Q$, Q is an $A_{l,r}$-loop. If Q/N is an abelian group, then $Q/Z(Q)$ is conjugacy closed.*

In fact, we have proved a somewhat stronger result as well:

Proposition 1.5.6 [CsD3, Proposition 3.2]**.** *Let Q be an $A_{r,l}$-loop in which the nucleus is normal and Q/N is an abelian group. If $[A,B] \leq \mathrm{Aut}\ Q$, then $Q/Z(Q)$ is a conjugacy closed loop.*

As the Buchsteiner loops satisfy these conditions we get

Corollary 1.5.7 [CsD3, Theorem 3.5]**.** *Let Q be a Buchsteiner loop. Then $Q/Z(Q)$ is a conjugacy closed loop.*

In Proposition 1.5.6 the requirement that Q is an $A_{r,l}$-loop seems to be too strong. In case of $[A,B] \leq \mathrm{Aut}\ Q$ we shall obtain an exact description when $Q/Z(Q)$ is conjugacy closed. For this aim we need the following subsets for a loop Q:

$$L_F(Q) = \{ {L_z}^{-1} {L_x}^{L_y} \mid \ {L_z}^{-1} {L_x}^{L_y} \in \mathrm{Inn}\ Q,\ x, y \in Q \},$$
$$R_F(Q) = \{ {R_w}^{-1} {R_x}^{R_y} \mid {R_w}^{-1} {R_x}^{R_y} \in \mathrm{Inn}\ Q,\ x, y \in Q \}.$$

In the following statements A_0, B_0 are defined as in Lemma 1.5.4.

Proposition 1.5.8 [Cs7, Proposition 3.7]**.** *Let Q be a loop such that $N \trianglelefteq Q$ and Q/N is an abelian group. Suppose $[A,B] \leq \mathrm{Aut}\ Q$. Then $Q/Z(Q)$ is a CC loop if and only if $L_F(Q) \subseteq \mathrm{Aut}\ Q$ and $R_F(Q) \subseteq \mathrm{Aut}\ Q$.*

Proof. Let $G = \mathrm{Mlt}\ Q$, $H = \mathrm{Inn}\ Q$.

Let $h^* \in L_F(Q)$ be arbitrary. Lemma 1.5.4 i), ii) give that there exist $\alpha_1, \alpha_2 \in A$ such that $\alpha_1^{\alpha_2} = \alpha_1 \delta h^*$ with $\delta \in A_0$. Then $\alpha_1^{\alpha_2^{-1}} = \alpha_1 ((h^*)^{-1})^{\alpha_2^{-1}} (\delta^{-1})^{\alpha_2^{-1}}$. Using $A_0 \trianglelefteq G$ (see Lemma 1.5.4 iv)) we get $\alpha_1^{\alpha_2^{-1}} = \alpha_1 \gamma ((h^*)^{-1})^{\alpha_2^{-1}}$ with $\gamma \in A_0$.

i) First suppose $L_F(Q) \subseteq \text{Aut } Q$ and $R_F(Q) \subseteq \text{Aut } Q$. Since $h^* \in L_F(Q)$ it follows $h^* \in \text{Aut } Q$, whence using Lemma 1.5.4 vi) we can conclude $((h^*)^{-1})^{\alpha_2^{-1}} = \gamma_0(h^*)^{-1}$ with $\gamma_0 \in A_0$ consequently $\alpha_1^{\alpha_2^{-1}} = \alpha_1\alpha_0(h^*)^{-1}$ with $\alpha_0 \in A_0$. Set $h = (h^*)^{-1}$, clearly $h \in \text{Aut } Q$. Let $\beta \in B$. We have ${\alpha_1}^\beta = \alpha_1 h_1$, $\beta^{\alpha_2} = \beta h_0$ with $h_1, h_0 \in [A, B]$. Then ${\alpha_1}^\beta = {\alpha_1}^{\beta^{\alpha_2 h_0^{-1}}} = \alpha_1 h_1$. Thus ${\alpha_1}^\beta = {\alpha_1}^{\alpha_2^{-1}\beta\alpha_2 h_0^{-1}} = (\alpha_1\alpha_0 h)^{\beta\alpha_2 h_0^{-1}}$. Using Lemma 1.5.4 vi) $h^\beta = h\beta_0$ holds with $\beta_0 \in B_0$. Hence ${\alpha_1}^\beta = (\alpha_1 h_1 \alpha_0 h \beta_0)^{\alpha_2 h_0^{-1}} = (\alpha_1 h_1^{h\alpha_2}\beta_0)^{h_0^{-1}}$. As $h_1, h_0 \in [A, B]$, Lemma 1.5.4 vii), v), vi) imply ${\alpha_1}^\beta = \alpha_1\alpha^* h_1 \alpha^{**}\beta_0 = \alpha_1 h_1$, where $\alpha^*, \alpha^{**} \in A_0$. Using Lemma 1.5.4 v), again we can conclude $\beta_0 \in B_0 \cap A_0$. Since $A_0 \cap B_0 \subseteq Z(G)$, whence $h^\beta \in hZ(G)$. As ${\alpha_1}^{\alpha_2^{-1}} = \alpha_1\alpha_0 h$ and $\beta \in B$ is arbitrary we get $h \in \text{core}_G Z(G)H$. Thus $Q/Z(Q)$ is left conjugacy closed. In a similar way $Q/Z(Q)$ is an RCC loop, consequently $Q/Z(Q)$ is a CC loop.

ii) Suppose $Q/Z(Q)$ is a CC loop. Then $R_F(Q) \cup L_F(Q) \subseteq \text{core}_G Z(G)H$. Since $h^* \in L_F(Q)$ we have $((h^*)^{-1}) \in \text{core}_G Z(G)H$, consequently $((h^*)^{-1})^{\alpha_2^{-1}} = (h^*)^{-1}\widetilde{h}z$ with $\widetilde{h} \in H$, $z \in Z(G) \cap (A_0 H)$. Thus $\alpha_1^{\alpha_2^{-1}} = \alpha_1\gamma z(h^*)^{-1}\widetilde{h}$. Put $\gamma z = \alpha_0 \in A_0$, $h = (h^*)^{-1}\widetilde{h}$, so $\alpha_1^{\alpha_2^{-1}} = \alpha_1\alpha_0 h$. Clearly $h \in \text{core}_G(Z(G))H$. Let $\beta \in B$, we have ${\alpha_1}^\beta = \alpha_1 h_1$, $\beta^{\alpha_2} = \beta h_0$ with $h_1, h_0 \in H \cap [A, B]$. Then ${\alpha_1}^\beta = {\alpha_1}^{\beta^{\alpha_2 h_0^{-1}}} = \alpha_1 h_1$. Let us use the same notation and repeat the steps of part i), then we get $\alpha_1^\beta = {\alpha_1}^{h_0^{-1}} {h_1}^{h\alpha_2 h_0^{-1}} (h^{-1}h^\beta)^{\alpha_2 h_0^{-1}}$. Since $h^{-1} \in \text{core}_G Z(G)H$, Lemma 1.5.4 vi) implies $h^\beta = hzh_2$ with $z \in Z(G)$, $h_2 \in H$, whence $(h^{-1}h^\beta)^{\alpha_2 h_0^{-1}} = (zh_2)^{\alpha_2 h_0^{-1}}$ with $\alpha_{01} \in A_0$. As $h_1 = {\alpha_1}^{-1}{\alpha_1}^\beta \in [A, B]$ it follows $h_1 \in \text{Aut } Q$, using Lemma 1.5.4 vii) we can conclude ${h_1}^h = h_1$, whence ${h_1}^{\alpha_2 h_0^{-1}} = (h_1\widetilde{\alpha})^{h_0^{-1}}$ with $\widetilde{\alpha} \in A_0$ by Lemma 1.5.4 vi). Hence ${\alpha_1}^\beta = \alpha_1\alpha_{01}h_1\widetilde{\alpha}{h_2}^{\alpha_2 h_0^{-1}} = \alpha_1 h_1$ with $\alpha_{01} \in A_0$. Since $A_0 \trianglelefteq G$ we can conclude $h_2 = e$, i.e. $h^\beta = hz$. As $\beta \in B$ is arbitrary we get $h \in \text{Aut } Q$. We have $\alpha_1^{\alpha_2^{-1}} = \alpha_1\alpha_0 h$, whence $\alpha_1^{\alpha_2} = \alpha_1(h^{-1})^{\alpha_2}(\alpha_0^{-1})^{\alpha_2}$. Using $A_0 \triangleleft G$ and $h^{-1} \in \text{Aut } Q \cap \text{core}_G(Z(G))H$ it follows $\alpha_1^{\alpha_2} = \alpha_1\xi z h^{-1}$ with $\xi \in A_0$. On the other hand we have $\alpha_1^{\alpha_2} = \alpha_1\delta h^*$, whence $h^* = h^{-1}$, and we can conclude $L_F(Q) \subseteq \text{Aut } Q$. We can show similarly $R_F(Q) \subseteq \text{Aut } Q$. $\square$

We give another sufficient condition for the conjugacy closedness of $Q/Z(Q)$.

Proposition 1.5.9 [Cs7, Proposition 3.8]**.** *Let Q be a loop such that $N \trianglelefteq Q$, Q/N is an abelian group. Suppose $L_F(Q) \cup R_F(Q) \subseteq Z(\text{Inn } Q)$. Then $Q/Z(Q)$ is a CC loop.*

Proof. Let $G = \text{Mlt } Q$ and $H = \text{Inn } Q$. We have $B_0 \leq C_G(\langle A \rangle)$ whence $B_0 \leq C_G(L_F(Q))$ whence $L_F(Q) \subseteq Z(H)$ implies $L_F(Q) \subseteq Z(B_0 H)$. Since $B_0 H \trianglelefteq G$ (see Lemma 1.5.4 i)) it follows $Z(B_0 H) \trianglelefteq G$. By Lemma 1.5.4 iii) $Z(B_0 H) = Z(G) \times (Z(B_0 H) \cap H)$ consequently $L_F(Q) \subseteq Z(B_0 H) \cap H \leq \text{core}_G Z(G)H$ i.e.

$Q/Z(Q)$ is left conjugacy closed. In a similar way we get that $Q/Z(Q)$ is an RCC loop too. $\square$

In case $[A, B] \leq \mathrm{Aut}\ Q$ the sufficient condition for the conjugacy closedness of $Q/Z(Q)$ in the previous proposition can be proved to be necessary.

Proposition 1.5.10 [Cs7, Proposition 3.9]**.** *Let Q be a loop such that $N \trianglelefteq Q$, Q/N is abelian group. Suppose that $[A, B] \leq \mathrm{Aut}\ Q$. Then $Q/Z(Q)$ is conjugacy closed if and only if $L_F(Q) \cup R_F(Q) \subseteq Z(\mathrm{Inn}\, Q)$.*

Proof. Let $G = \mathrm{Mlt}\ Q$, $H = \mathrm{Inn}\ Q$.

i) Suppose first $L_F(Q) \cup R_F(Q) \subseteq Z(\mathrm{Inn}\ Q)$. Then Proposition 1.5.9 implies our statement.

ii) Suppose $Q/Z(Q)$ is a CC loop. Let $\alpha_1, \alpha_2 \in A$. Then using Lemma 1.5.4 ii) we get ${\alpha_1}^{\alpha_2} = \alpha_1 \alpha_0 h$ with $\alpha_0 \in A_0$, $h \in H \cap \langle A \rangle$. Clearly $h \in L_F(Q)$, since $L_F(Q) \subseteq \mathrm{Aut}\ Q$ by Proposition 1.5.8, it follows $h^a \in hA_0$ for every $a \in A$ (see Lemma 1.5.4 vi)). The conjugacy closedness of $Q/Z(Q)$ implies $h \in \mathrm{core}_G Z(G)H$, whence $h^a \in hZ(G)$. Similarly $h^b \in hZ(G)$ for every $b \in B$. As $G = \langle A, B \rangle$ we can conclude $h \in Z(\mathrm{Inn}\ Q)$, whence clearly $L_F(Q) \subseteq Z(\mathrm{Inn}\ Q)$. In a similar way we get $R_F(Q) \subseteq Z(\mathrm{Inn}\ Q)$. $\square$

In case of Buchsteiner loops we have a necessary and sufficient condition that $Q/Z(Q)$ is a group:

Proposition 1.5.11 [DK, Lemma 7.2]**.** *Let Q be a Buchsteiner loop. Then $Q/Z(Q)$ is a group, i.e. $A(Q) \leq Z(Q)$ if and only if $[A, B] \leq Z(\mathrm{Inn}\ Q)$.*

We generalize this result in the following way:

Proposition 1.5.12 [Cs7, Proposition 3.11]**.** *Let Q be a loop such that $N \trianglelefteq Q$, Q/N is an abelian group and $[A, B] \leq Z(\mathrm{Inn}\ Q)$. Then $Q/Z(Q)$ is a group i.e. $A(Q) \leq Z(Q)$.*

Proof. Let $G = \mathrm{Mlt}\ Q$, $H = \mathrm{Inn}\ Q$. By Lemma 1.5.4 ii) G/A_0H is abelian. We show $[A, B] \leq Z(A_0H)$. By Lemma 1.5.4 v) $A_0 \leq C_G([A, B])$. The condition $[A, B] \leq Z(H)$ implies $[A, B] \leq Z(A_0H)$. Since $Z(A_0H) = (Z(G) \cap A_0) \times (Z(A_0H) \cap H)$ and $A_0H \trianglelefteq G$, by Lemma 1.5.4 iii), ii) it follows $Z(A_0H) \trianglelefteq G$. Thus we get $Z(A_0H) \leq \mathrm{core}_G Z(G)H$, consequently $Q/Z(Q)$ is a group. $\square$

In case $[A, B] \leq \mathrm{Aut}\ Q$ the above mentioned sufficient condition can be proved to be necessary.

Proposition 1.5.13 [Cs7, Proposition 3.12]**.** *Let Q be a loop such that $N \trianglelefteq Q$, Q/N is an abelian group and $[A,B] \leq \mathrm{Aut}\ Q$. Then $Q/Z(Q)$ is a group if and only if $[A,B] \leq Z(\mathrm{Inn}\ Q)$.*

Proof. Let $G = \mathrm{Mlt}\ Q$ and $H = \mathrm{Inn}\ Q$.

i) First suppose $[A,B] \leq Z(\mathrm{Inn}\ Q)$. Then by Proposition 1.5.12 it follows our statement.

ii) Suppose $Q/Z(Q)$ is a group. Then $[A,B] \leq \mathrm{core}_G Z(G)H$. Since $[A,B] \leq \mathrm{Aut}\ Q \cap H$ using Lemma 1.5.4 v) $t^a \in tA_0$ for every $t \in [A,B]$ and $a \in A$. Consequently $t^a \in tZ(G)$ holds. Similarly we get $t^b \in tZ(G)$ for every $b \in B$. As $G = \langle A,B \rangle$ we can conclude $t \in Z(H)$, i.e. $[A,B] \leq Z(H)$. □

In the following we study the case of abelian inner mapping group.

Proposition 1.5.14 [Cs7, Proposition 3.13]**.** *Let Q be a Buchsteiner loop with abelian inner mapping group. Then*

i) *$Q/Z(Q)$ is a group.*

ii) *Q is nilpotent of class at most three.*

Proof. i) See Proposition 1.5.11.

ii) Since a CC loop with abelian inner mapping group is nilpotent of class at most two [CsD1, Proposition 2.5] our statement follows. □

We analyze the general case:

Theorem 1.5.15 [Cs7, Theorem 3.14]**.** *Let Q be a loop with abelian inner mapping group such that $N \trianglelefteq Q$ and Q/N is an abelian group. Then the following statements are true:*

i) *$Q/Z(Q)$ is a group.*

ii) *Q is nilpotent of class at most three.*

Proof. i) See Proposition 1.5.12.

ii) See the proof of Proposition 1.5.14 ii). □

In case of abelian inner mapping group under the conditions of Proposition 1.5.6 we can prove more, namely the nilpotency of class at most three of the multiplication group.

For this aim we need the following

Lemma 1.5.16 [Cs7, Lemma 3.15]**.** *Let Q be a loop with abelian inner mapping group such that $N \trianglelefteq Q$ and Q/N is an abelian group. Let $G = \mathrm{Mlt}\ Q$, $H = \mathrm{Inn}\ Q$, and G_0 is the normal closure of H in G. Then*

i) $h^a \in h(Z(G) \cap G_0)$, $h^b \in h(Z(G) \cap G_0)$ *for every* $h \in H \cap \mathrm{Aut}\ Q$ *and* $a \in A$, $b \in B$.

ii) ${a_1}^a \in a_1(Z(G) \cap G_0)$ *for every* $a_1 \in A_0 \cap G_0$, $a \in A$.

Proof. i) Let $h \in H \cap \mathrm{Aut}\ Q$. Then $h^{a^{-1}} \in hA_0$ by Lemma 1.5.4 vi). Since $A_0 \trianglelefteq G$ (see Lemma 1.5.4 iv)) and $G_0 \trianglelefteq G$ we get $h^{a^{-1}} \in hA_0 \cap G_0$. Clearly $A_0H \geq G_0$, whence $G_0 = (A_0 \cap G_0)H$, consequently $h^{a^{-1}} \in h(A_0 \cap G_0)$. Let $b \in B \cap aH$, in a similar way we can show $h^{b^{-1}} \in h(B_0 \cap G_0)$. The commutativity of H implies $h^{a^{-1}b} = h$, whence $h^{a^{-1}} = h^{b^{-1}} \in h(A_0 \cap B_0)$. Since $A_0 \cap B_0 \subseteq Z(G)$ we get $h^a \in h(Z(G) \cap G_0)$, $h^b \in h(Z(G) \cap G_0)$ for every $a \in A$, $b \in B$.

ii) By Theorem 1.5.15 we have $Q/Z(Q)$ is of nilpotency class at most two. Hence clearly $Q/Z(Q)/Z(Q/Z(Q))$ is an abelian group. Let $U = \mathrm{core}_G Z(G)H$. Then $\mathrm{Mlt}\,(Q/Z(Q)) = \mathrm{Mlt}\ Q/U$. Let Z^* be the inverse image of $Z(\mathrm{Mlt}\ Q/U)$. Using $Q/Z(Q)/Z(Q/Z(Q))$ is an abelian group it follows $Z^*\mathrm{Inn}\ Q \trianglelefteq \mathrm{Mlt}\ Q$. By Proposition 1.2.8 iii) $G_0 \leq Z^*\mathrm{Inn}\ Q$. Applying $Z(\mathrm{Mlt}\ Q) \subseteq A \cap B$ for $Z(\mathrm{Mlt}\ Q/Z(Q))$ we get $b_1^{-1}a_1 \in U \cap H$ for every $a_1 \in G_0 \cap A_0$, $b_1 \in B \cap a_1H$. Thus $(b_1^{-1}a_1)^a \in b_1^{-1}a_1U$ for every $a \in A$. We have $b_1 \in G_0 \cap B_0 \leq C_G(A)$, whence ${a_1}^a \in a_1U$. Since $U = \mathrm{core}_G Z(G)H$ it follows ${a_1}^a = a_1z_1h_1$ with $z_1 \in Z(G)$, $h_1 \in U \cap H$. As $A_0 \trianglelefteq G$ (see Lemma 1.5.4 iv)) ${a_1}^a = a_1z_1$ holds. $G_0 \trianglelefteq G$ implies $z_1 \in Z(G) \cap G_0$. □

We return to our statement.

Theorem 1.5.17 [Cs7, Theorem 3.16]**.** *Let Q be an $A_{r,l}$-loop with abelian* $\mathrm{Inn}\ Q$ *such that $N \trianglelefteq Q$, Q/N is an abelian group. Suppose $[A,B] \leq \mathrm{Aut}\,Q$. Then Q and* $\mathrm{Mlt}\ Q$ *are nilpotent of class at most three.*

Proof. By Theorem 1.5.15 Q is nilpotent of class at most three.

Let $G = \mathrm{Mlt}\ Q$, $H = \mathrm{Inn}\ Q$. Let $M = \langle A\rangle\,[A,B]$. We show $M \trianglelefteq G$. Using $[A,B] \leq \mathrm{Aut}\ Q$ and H is abelian Lemma 1.5.16 i) implies $Z(G)[A,B] \trianglelefteq G$. Since $Z(G) \leq \langle A\rangle$ it follows $M \leq G$. We have $\langle A\rangle \cap H \leq \mathrm{Aut}\ Q$, $[A,B] \leq \mathrm{Aut}\ Q$, using $\langle A,B\rangle = G$ and Lemma 1.5.16 i) we get $M \trianglelefteq G$.

Let $Z_1 = Z(G) \cap G_0$ (G_0 is the normal closure of H in G), $D = G_0 \cap M$ and $A_1 = G_0 \cap A_0$. We show $D/Z_1 \leq Z(G/Z_1)$. Using Lemma 1.5.16 ii), $A_1 \leq C_G(B)$ and $G = \langle A,B\rangle$ we can conclude $A_1Z_1/Z_1 \leq Z(G/Z_1)$. As $D \cap H = (\langle A\rangle \cap H)[A,B]$ and $(\langle A\rangle \cap H)[A,B] \leq \mathrm{Aut}\,Q$, Lemma 1.5.16 i) implies $D/Z_1 \leq Z(G/Z_1)$. Since $G/M \cong H/H \cap M$ it follows G/M is abelian. Using G/G_0 is abelian too we get $G' \leq M \cap G_0 = D$, consequently G/D is abelian. Thus G is nilpotent of class at most three. □

Using the previous result we describe the structure of Buchsteiner loop with abelian inner mapping groups. For this aim we need the following:

Proposition 1.5.18 [CsD3 Lemma 7.2, Proposition 7.3]**.** *If Q is a Buchsteiner loop with abelian inner mapping group, then Q/N is an elementary abelian 2-group.*

Corollary 1.5.19 [Cs7, Corollary 3.18]**.** *Let Q be a Buchsteiner with abelian* $\mathrm{Inn}\, Q$*, let $A_0 = \{L_c \mid c \in N\}$. Then* $\mathrm{Mlt}\ Q/A_0\mathrm{Inn}\ Q$ *is elementary abelian 2-group.*

Proof. The structure of the multiplication group of the factorloop and Proposition 1.5.18 imply this statement. □

Proposition 1.5.20 [Cs7, Proposition 3.19]**.** *Let Q be a Buchsteiner loop with abelian* $\mathrm{Inn}\ Q$*. Then the following statements are true:*

i) *Q and* $\mathrm{Mlt}\ Q$ *are nilpotent of class at most three.*

ii) *$aa^b \in A_0$ for every $a \in A$, $b \in B \cap a\,\mathrm{Inn}\ Q$ where $A_0 = \{L_c \mid c \in N\}$.*

iii) *$\langle A \rangle \cap \mathrm{Inn}\ Q$ is an elementary abelian 2-group.*

Proof. i) By Theorem 1.5.15 Q is nilpotent of class at most three. We have Q is an $A_{r,l}$-loop, $N \trianglelefteq Q$, Q/N is an abelian group, $[A, B] = \langle A \rangle \cap \mathrm{Inn}\ Q = \langle B \rangle \cap \mathrm{Inn}\, Q$ (see Proposition 1.5.3) whence $[A, B] \leq \mathrm{Aut}\, Q$, so we can apply Theorem 1.5.17.

ii) See the definition of Buchsteiner loops and Corollary 1.5.19.

iii) By ii) $aa^b \in A_0$ for every $a \in A$, $b \in B \cap a\,\mathrm{Inn}\ Q$. Let $b_1 \in B$ be arbitrary. Clearly $a^b = ah$, $a^{b_1} = ah_1$ with $h, h_1 \in \mathrm{Inn}\ Q$, whence $aa^b = (aa^b)^{b_1} = ah_1ah_1h^b = a^2{h_1}^a h_1 h^b$. Lemma 1.5.16 i) implies ${h_1}^a = h_1 z_1$, $h^b = hz$ with $z_1 z \in Z(\mathrm{Mlt}\ Q)$. Hence $(aa^b)^{b_1} = a^2 h {h_1}^2 z_1 z = aah$. Thus ${h_1}^2 = e$. □

We give a characterization theorem about the Buchsteiner loops with abelian inner mapping group.

Theorem 1.5.21 [Cs7, Theorem 3.20]**.** *Let Q be a finite loop. Then Q is a Buchsteiner loop with abelian inner mapping group if and only if $Q = Q_1 \times Q_2$, where Q_1 is a Buchsteiner loop with abelian inner mapping of order 2^t and Q_2 is a group of odd order with abelian inner mapping group. Additionally* $\mathrm{Mlt}\ Q = \mathrm{Mlt}\ Q_1 \times \mathrm{Mlt}\ Q_2$ *where* $\mathrm{Mlt}\ Q_1 \in \mathrm{Syl}_2(\mathrm{Mlt}\ Q)$.

Proof. i) Clearly, if $Q = Q_1 \times Q_2$ with the given properties, then Q is a Buchsteiner loop with abelian $\mathrm{Inn}\ Q$.

ii) Conversely suppose Q is a Buchsteiner loop with abelian $\mathrm{Inn}\ Q$. Let $G = \mathrm{Mlt}\ Q$, $H = \mathrm{Inn}\ Q$. By Proposition 1.5.20 i) G is nilpotent of class at most three. So $G = S \times T$, where $S \in \mathrm{Syl}_2(G)$.

First we show $S = (S \cap A)(S \cap H)$. Let $S_1 \in \mathrm{Syl}_2(\langle A \rangle)$. Since $\langle A \rangle \trianglelefteq G$ (see Proposition 1.5.3) it follows $S_1 \trianglelefteq G$. Let $S_0 \in \mathrm{Syl}_2(H)$, then $S_1 S_0 \leq G$. Using $\langle A \rangle H = G$ we can conclude $S_1 S_0 = S \in \mathrm{Syl}_2(G)$. As $\langle A \rangle \cap H$ is elementary abelian 2-group (see Proposition 1.5.20 iii)). $S \geq \langle A \rangle \cap H$ holds, whence $S_1 = S \cap \langle A \rangle =$

$(S\cap A)(\langle A\rangle\cap H)$. We have $S=S_1S_0$, consequently $S=(S\cap A)(S\cap H)$. In a similar way $S=(S\cap B)(S\cap H)$.

Since G/A_0H is elementary abelian 2-group (see Corollary 1.5.19) and $S\geq\langle A\rangle\cap H$ we can conclude $T\leq A_0H$ and $T_1=T\cap\langle A\rangle\leq A_0$. As G is nilpotent and T is a Hall subgroup of G, by Hall's theorems $T_1\trianglelefteq G$. We have $H=S_0\times T_0$ where $T_0\leq T$ is a Hall subgroup of H. Using $T\leq A_0H$, $A_0\cap H=1$ and $A_0\trianglelefteq G$ it follows $T=T_1\cdot T_0=(T\cap\langle A_0\rangle)(T\cap H)$. Similarly $T=(T\cap B_0)(T\cap H)$.

Let

$$\begin{aligned} A_S &= S\cap A \quad & A_T &= T\cap A_0,\\ B_S &= S\cap B \quad & B_T &= T\cap B_0. \end{aligned}$$

Since $A_0A=A$, $B_0B=B$ we have $A_TA_S\subseteq A$, $B_TB_S\subseteq B$. Clearly $|A_TA_S|=|A_T|\,|A_S|$, $|B_TB_S|=|B_T|\,|B_S|$. Using $G=S\times T$, $S=(S\cap A)(S\cap H)=(S\cap B)(S\cap H)$, $T=(T\cap A_0)(T\cap H)=(T\cap B_0)(T\cap H)$ we get $A=A_TA_S$, $B=B_TB_S$.

Let

$$\begin{aligned} Q_1 &= \{c\in Q \mid L_c\in S\},\\ Q_2 &= \{d\in Q \mid L_d\in T\}. \end{aligned}$$

As $SH\leq G$ and $TH\leq G$ we can conclude Q_1 and Q_2 are normal subloops of Q. We show Mlt $Q_1=S$ and Mlt $Q_2=T$. By Niemenmaa and Kepka's theorem [NK1] it is enough to show:

i_1) $\langle S\cap A, S\cap B\rangle=S$, i_2) $\mathrm{core}_S(S\cap H)=1$, i_3) $[S\cap A, S\cap B]\leq S\cap H$,
j_1) $\langle T\cap A_0, T\cap B_0\rangle=T$, j_2) $\mathrm{core}_T(T\cap H)=1$, j_3) $[T\cap A_0, T\cap B_0]\leq T\cap H$.

We have $\langle A,B\rangle=G$, since $A=A_TA_S$, $B=B_TB_S$ and $G=S\times T$ i_1) and j_1) are true.

$G=S\times T$ implies $\mathrm{core}_S(S\cap H)\leq S\cap C_G(T)$ and $\mathrm{core}_S(S\cap H)\trianglelefteq G$. Using $\mathrm{core}_G H=1$ we get $\mathrm{core}_S(S\cap H)=1$. In a similar way j_2) follows.

i_3) and j_3) are the consequences of $[A,B]\leq H$.

Clearly Q_1 is a Buchsteiner loop of order 2^t with abelian Inn Q_1 $(=S\cap H)$.

Since Mlt $Q_2=(T\cap A_0)(T\cap H)$, $T\cap\langle A\rangle\leq A_0$, $A_0\trianglelefteq$ Mlt Q it follows that Q_2 is a group of odd order with abelian Inn Q_2 $(=T\cap H)$. □

1.6. On the solvability of loops and groups

As defined in the introduction a loop Q is solvable if it has a series $1 \subseteq Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_n = Q$, where Q_{i-1} is normal subloop in Q_i and the factorloops Q_i/Q_{i-1} are abelian groups for every $1 \leq i \leq u$.

Bruck in [Br] showed if a loop Q is nilpotent, then the multiplication group $\operatorname{Mlt} Q$ is solvable. The converse is not true: let Q be the dihedral group of order 6, when $\operatorname{Mlt} Q$ is solvable being of order 36, but Q is not centrally nilpotent. In 1996 Vesanen [Ve1, Theorem 1] proved if Q is a finite loop, then the solvability of the multiplication group of Q implies the solvability of loop Q. Vesanen's result is fundamental and very deep because it opens a large variety of possibilities to create solvability criteria for finite loops.

Problem. Which properties of the inner mapping group $\operatorname{Inn} Q$ guarantee the solvability of the multiplication group $\operatorname{Mlt} Q$?

Kepka and Niemenmaa started to study this solvability question. First they showed in [NK3, Theorem 3.4] that the solvability of the multiplication group $\operatorname{Mlt} Q$ also follows provided $\operatorname{Inn} Q$ is finite abelian, in this case even Q is centrally nilpotent. Later Niemenmaa started to investigate the non-abelian case and using this result [N5, Theorem 3.4] we know that a finite loop Q is solvable, if the order of the inner mapping group is six. His next result [N7, Theorem 4.2] is that if the inner mapping group is a dihedral 2-group, then the multiplication group of the loop is solvable and the loop is centrally nilpotent.

The more general problem is that case if the order of the inner mapping group is the product of two different primes p and q. Niemenmaa could show [N6, Theorem 4.3] by using the classification of finite simple groups that the loop Q is solvable, if Q is finite, the order of the inner mapping group is pq where p and q are different primes such that $q = 2$ and $p \leq 61$, $q = 3$ and $p \leq 31$, $q = 5$ and $p \leq 11$. Then Myllylä and Niemenmaa [MN, Theorem 2.3] proved the solvability of a commutative loop in case the order of the inner mapping group is $2p$ where $p = 4t + 3$ is an odd prime.

In [CsN1] with M. Niemenmaa we have managed to show a more general case: the solvability of the multiplication group in case the inner mapping group has order $2p$, where p is an odd prime.

Extending the examinations to dihedral inner mapping groups with M. Niemenmaa and K. Myllylä [CsMN] we were able to show that if the inner mapping is a dihedral group of order $2p^n$ (p is an odd prime), then the multiplication group is solvable.

Later K. Myllylä [M, Theorem 4.3] generalized this result for dihedral group of order $2n$ where n is an odd natural number.

Studying the most general case, when the order of inner mapping group is the product of two different primes with Niemenmaa we obtained the following [CsN2] if the inner mapping group of a finite loop Q is of order pq, where p and q are odd primes satisfying $p = 2q^m + 1$ then Q is a solvable loop.

Finally in 2002 A. Drápal could prove the most general case [D2, Corollary 4.7], namely the solvability of a finite loop Q, if the order of its nonabelian inner mapping group is the product of two different primes. He was working not by using connected transversals but studying the multiplication group as permutations and the loop structure with special attention to the orbits of inner mapping group.

We prove our results by applying the theory of connected transversals which are very useful tool when studying the structure of loops. The point is that the structure of the multiplication group reflects the structural properties of the corresponding loop. This approach to the structural properties of loop has been very fruitful, in particular, in the case of nilpotent and solvable groups.

Solvability in $2p$ case

Our aim in this section is to prove the solvability of the multiplication group, if the inner mapping group has order $2p$ (p is an odd prime); moreover in case of finite loop to show the solvability of the loop.

Many properties of loops can be reduced to the properties of connected transversals in the multiplication group. It is clear that also here we have to apply the theory of connected transversals. First we give basic information about connected transversals together with some preliminary lemmas which are needed later. Then we prove our main theorem which is formulated in pure group theoretic terms: If a group G has a subgroup H of order $2p$ (p is an odd prime) and if there exists a pair of H-connected transversals in G, then G is solvable. Our proof uses rather elementary counting arguments and a detailed analysis on how certain involutions of G are distributed among the cosets of H. Finally, we apply our results on loop theory and we get a solvability criterion for finite loops.

Some lemmas

Let us recall: Let G be a group, $H \leq G$ and let A and B be two left transversals to H in G. We say that the two transversals A and B are H-connected if $a^{-1}b^{-1}ab \in H$ for every $a \in A$ and for every $b \in B$. By $L_G(H)$ we denote the core of H in G (the largest normal subgroup of G contained in H). If Q is a loop then $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$ are $\operatorname{Inn} Q$-connected transversals in $\operatorname{Mlt} Q$ and the core of $\operatorname{Inn} Q$ in $\operatorname{Mlt} Q$ is trivial.

In the following three lemmas we introduce some basic results on connected transversals. We assume that G is a group, $H \leq G$ and A, B are H-connected transversals in G.

Lemma 1.6.1 [CsN1, Lemma 1.2]**.** *If $L_G(H) = 1$, then $1 \in A \cap B$ and $N_G(H) = H \times Z(G)$.*

Lemma 1.6.2 [CsN1, Lemma 1.3]**.** *Now A and B are left (and right) transversals to H^g for every $g \in G$.*

Lemma 1.6.3 [CsN1, Lemma 1.4]**.** *If H is cyclic, then G is solvable.*

For the proofs, see [KN1], [NK1, p. 113–114].

We also need the following group theoretic results for the proof of our main theorem.

Lemma 1.6.4 [CsN1, Lemma 1.5]. *Let G be a transitive permutation group on the finite set X and for $g \in G$ denote $F(g) = \{x \in X : g(x) = x\}$. Then $|G| = \sum_{g \in G} |F(g)|$.*

For the proof, see [NST, Theorem 9.1].

Lemma 1.6.5 [CsN1, Lemma 1.6]. *Let G be a finite group and let $G = AH$ where A is abelian and H is a subgroup such that $|H| = pq$ where $p > q$ are prime numbers. Then G is solvable.*

For the proof, see [N6, Lemma 2.5].

Main Theorem

In this section we consider the situation that G is a group, $H \leq G$, $|H| = 2p$ where p is an odd prime number and A, B are H-connected transversals in G. We denote by P the Sylow p-subgroup of H. We shall start with a series of preliminary lemmas which are needed later.

Lemma 1.6.6 [CsN1, Lemma 2.1]. *If H is a maximal subgroup of G and $L_G(H) = 1$, then $|H \cap H^g| \leq 2$ for every $g \in G \setminus H$.*

Proof. Clearly, $N_G(H) = H$. Thus $H \neq H^g$, whenever $g \in G \setminus H$. If $|H \cap H^g| = p$, then $P \leq H \cap H^g$. Now P is a normal subgroup of H and H^g, hence P is normal in $\langle H, H^g \rangle = G$. Since $L_G(H) = 1$, this is not possible. Thus either $|H \cap H^g| = 1$ or $|H \cap H^g| = 2$ for every $g \in G \setminus H$. □

Lemma 1.6.7 [CsN1, Lemma 2.2]. *Let H be a maximal subgroup of G and $L_G(H) = 1$. If $a \in A$ and $b \in B$ such that $aH = bH$, then $b^{-1}a = a^{-1}b \in H \cap H^{a^{-1}} = H \cap H^{b^{-1}}$.*

Proof. Now $a^{-1}b^{-1}ab \in H$ and $b \in aH$. Then $b^{-1}a \in H \cap H^{a^{-1}} = H \cap H^{b^{-1}}$. From Lemma 1.6.6 it follows that either $b^{-1}a$ is an involution or $b^{-1}a = 1$. In both cases $b^{-1}a = a^{-1}b$. □

Lemma 1.6.8 [CsN1, Lemma 2.3]. *Let G be a finite group, H a maximal subgroup of G and $L_G(H) = 1$. If $H \cap H^a = 1$ for some $1 \neq a \in A$, then $A = B$and G is solvable.*

Proof. Let $H \cap H^a = 1$ and $aH = bH$, where $a \in A$ and $b \in B$. Clearly, $H \cap H^{a^{-1}} = 1$ and from Lemma 1.6.7 it follows that $a = b$. Then let $uH = wH$ where $u \in A$ and $w \in B$. Now $a \in A \cap B$, hence $a^{-1}u^{-1}au \in H$. Thus $a^{-1}u^{-1}aw \in H$ and finally $a^{-1}u^{-1}wa \in H$. It follows that $u^{-1}w \in H \cap H^{a^{-1}} = 1$, hence $u = w$. We conclude that $A = B$.

Then let $b \in a^2H \cap A$. Now $a^{-1}baH = bH = a^2H$ and thus $b^a \in a^2H$. It follows that $(b^{-1}a^2)^a = (b^{-1})^a a^2 \in H$ and $b^{-1}a^2 \in H \cap H^{a^{-1}} = 1$. Thus $b = a^2$. Now assume that $e \in A$. Then $e^{-1}e^{a^2}$ and $e^{-1}e^a$ are elements of H and since $(e^{-1}e^a)^a = (e^{-1})^a e^{a^2}$, we conclude that $e^{-1}e^a \in H \cap H^{a^{-1}} = 1$. Thus $e \in C_G(a)$ and this shows that, in fact, $A \subseteq C_G(a)$. Since $C_G(a) \cap H = 1$, we have $A = C_G(a)$.

Thus A is a subgroup of G and since $[A, A] \leq A \cap H = 1$, it follows that A is an abelian group. Thus $G = AH$ and by Lemma 1.6.5, G is solvable. The proof is complete. $\square$

We are now ready to prove

Theorem 1.6.9 [CsN1, Theorem 2.4]**.** *Let G be a group, $H \leq G$ and $|H| = 2p$, where p is an odd prime number. If there exist H-connected transversals A and B in G, then G is solvable.*

Proof. We first assume that G is finite and our proof is by induction on the order of G. As in [N6, proof of Theorem 3.1] we may assume that H is a maximal subgroup of G, $L_G(H) = 1$ and G is a simple group.

If there exists $1 \neq a \in A$ such that $H \cap H^a = 1$, then G is solvable by Lemma 1.6.8. Thus we may assume that $H \cap H^a > 1$ whenever $1 \neq a \in A$. Now $N_G(P) = H$ and thus P is a Sylow p-subgroup of G, hence $[G : H] = 1 + kp$. Since G is a simple group, k is an odd number. We can consider G as a permutation group acting on the set with $1 + kp$ points and H is a one point stabilizer. Since $H \cap H^g > 1$ whenever $g \in G \setminus H$, we conclude that in the action of H on the remaining kp points every orbit has length p. From Lemma 1.6.4 it follows that if $c \in H$ is an involution, then c is a product of $k(p-1)/2$ distinct transpositions. Thus c (and all conjugates of c) fix $k + 1$ points. We again apply the counting argument from Lemma 1.6.4 and we get

$$|G| = 1 + kp + (1 + kp)(p - 1) + N(1 + k),$$

where N is the number of conjugates of c in G. It follows that $N = (1+kp)p/(k+1)$ and thus $|C_G(c)| = 2(k + 1)$. As p does not divide $k + 1$, we conclude that $k + 1$ divides $1 + kp = (1 + k) + k(p - 1)$. Thus $k + 1$ divides $p - 1$.

The number of those conjugates of the involution c which are not contained in H is $N - p = kp(p - 1)/(k + 1)$. If every coset aH (here $1 \neq a \in A$) contains at most one conjugate of c, then $kp(p - 1)/(k + 1) \leq kp$. Thus $p - 1 \leq k + 1$ and, in fact, $p - 1 = k + 1$. It follows that every coset aH $(1 \neq a \in A)$ contains exactly one conjugate of c.

Now assume that $d = c^x \in aH$ where $a \in C_G(c)H - H$ and $x \in G$. Clearly, $c^d \in H$ and thus $c \in H^{d^{-1}} = H^d$ (d is an involution). We conclude that c and c^d are elements of $H \cap H^d$ and by Lemma 1.6.6, $c = c^d$. Thus $d \in C_G(c)$ and dc

is an involution. Since $|C_G(c)H| = (p-1)|H|$ and each coset $aH \neq H$ contains exactly one conjugate of c, it follows that $C_G(c)$ has $2(p-2)+1$ elements which are involutions. As $|C_G(c)| = 2(p-1)$, we conclude that $|C_G(c)| = 2^s (s \geq 2)$. But then $1 + kp = 1 + (p-2)p = (p-1)^2 = 2^{2s-2}$ and $|G| = 2^{2s-1}p$. Now it is clear that G is a solvable group.

Thus we may assume that there exists a coset $aH \neq H$ which contains at least two conjugates of c. Is it possible that aH contains more than two conjugates of c? If c^x, c^y and c^z are three different conjugates of c contained in aH, then $c^x = ah, c^y = ak$ and $c^z = at$, where $h, k, t \in H$. Now $c^x = (c^x)^{-1} = h^{-1}a^{-1}$ and $c^x c^y = h^{-1}k \in H$. As $c^x a^{-1} \in H^{a^{-1}}$ and $c^y a^{-1} \in H^{a^{-1}}$, we get $c^x c^y \in H \cap H^{a^{-1}}$ and by Lemma 1.6.6 we know that $c^x c^y \in H$ is an involution. Likewise $c^x c^z \in H \cap H^{a^{-1}}$ is an involution and now it is obvious that $c^x c^y = c^x c^z$. But then $c^y = c^z$, a contradiction.

Thus we may assume that there exists a coset $aH \neq H$ which contains exactly two conjugates of c. We denote these two different conjugates by c^x and c^y. Since $c^x, c^y \in aH$, it follows that $c^x a \in H$ and $c^y a \in H$. As $c^x a a^{-1} c^y = c^x c^y \in H$ and $c^x c^y$ is an involution, we conclude that either $c^x a$ or $c^y a$ is an involution. In what follows we assume that $c^x a = t \in H$ is an involution. Now we write $c^x c^y = d \in H$ and, of course, d is an involution and c^x, c^y and d commute with each other. If $t = d$, then $a = c^y$, but this is not possible by Lemma 1.6.2. Thus assume that $t \neq d$.

Now we write $E = C_G(t)H \cap B$. If $1 \neq b \in E$, then $a^b \in aH$ and $c^{xb} = (c^x)^b = (at)^b \in aH$. Since aH contains exactly two conjugates of c, it follows that either $(c^x)^b = c^x$ or $(c^x)^b = c^y$.

If $(c^x)^b = c^x$ for every $b \in E$, then applying Lemma 1.6.2 we see that E, Ec^x, Ec^y and Ed are disjoint subsets of $C_G(c^x)$. As $|E| = k+1$ and $|C_G(c^x)| = 2(k+1)$, we have too many elements in the subgroup $C_G(c^x)$.

Then assume that $(c^x)^b = c^x$ and $(c^x)^f = c^y$ for some $b, f \in E$. Now $c^x = h^{m^{-1}}$, where $h \in H$ and $m \in A$. Thus $b^m \in C_G(h)$ and also $b^m \in bH \subseteq C_G(t)H$. We conclude that $h = t$. Now $c^y = t^{m^{-1}f}$ and $t^{m^{-1}f} \in C_G(c^x) = C_G(t)^{m^{-1}}$, hence $t^{m^{-1}fm} \in C_G(t)$. Thus $m^{-1}fm \in C_G(h)$ and $f \in C_G(c^x)$. But then $(c^x)^f = c^x$, a contradiction.

Thus we may assume that $(c^x)^b = c^y$ for every $1 \neq b \in E$. Now $(c^y)^{gb} = c^y$, where $g = y^{-1}x$. It follows that $gb \in C_G(c^y)$ for every $1 \neq b \in E$. If we write $W = \{gb : 1 \neq b \in E\}$, then W, Wc^x, Wc^y and Wd are disjoint subsets of $C_G(c^y)$. Since $|W| = k$ and $|C_G(c^y)| = 2(k+1)$, we have $4k \leq 2(k+1)$ and thus $k = 1$. Now H has $1+p$ left cosets in$G$, H contains p conjugates of c and each of the cosets $aH \neq H$ contains at most two conjugates of c. Thus $(1+p)p/2 \leq p + 2p$ which means that $p = 3$ or $p = 5$. By [N6, Theorem 3.1], G is solvable.

Then assume that G is infinite. If $L_G(H) > 1$, then $H/L_G(H)$ is cyclic and $G/L_G(H)$ is solvable by Lemma 1.6.3 and clearly G is solvable.

Thus we can assume that $L_G(H) = 1$. Let $G = \langle A, B\rangle$ and let a and h be fixed elements from A and H. We write $F(a,h) = \{b \in B : a^{-1}b^{-1}ab = h\}$. If $b, c \in F(a,h)$, then $bc^{-1} \in C_G(a)$ and $b \in C_G(a)c$. Thus $F(a,h) \subseteq C_G(a)b_h$, where b_h is a fixed element from $F(a,h)$. Now $B = \cup_{h\in H} F(a,h)$ and $G = BH \subseteq C_G(a)\{b_h : h \in H\}H$. It follows that $[G : C_G(a)] \leq |H|^2$. Thus $[G : C_G(H)]$ is finite and therefore $[G : N_G(H)]$ is finite. By Lemma 1.6.1, $N_G(H) = H \times Z(G)$ and thus $[G : Z(G)]$ is finite. Since $|HZ(G)/Z(G)| = 2p$ we conclude that $G/Z(G)$ is solvable by the first part of our proof. But this means that G is solvable.

Then let $K = \langle A, B\rangle$ be a proper subgroup of G. Now A and B are $K \cap H$-connected transversals in K and thus it is clear that K is solvable. Since $[G : K]$ is finite, we conclude that $[G : L_G(K)]$ is finite. Now $HL_G(K)/L_G(K)$ is cyclic or of order $2p$ and this implies the solvability of $G/L_G(K)$. From the solvability of $L_G(K)$ it follows that G is solvable. The proof is complete.

Loop theoretical result

The relation between multiplication groups of loops and connected transversals was given in Theorem 1.1.1. The following solvability criterion has been proved by Vesanen [Ve1, Theorem 1].

Theorem 1.6.10. *If Q is a finite loop whose multiplication group is solvable, then Q is a solvable loop.*

By combining Theorem 1.6.10 with Theorems 1.1.1 and 1.6.9 we immediately have

Theorem 1.6.11 [CsN1, Theorem 3.2]**.** *If Q is a loop whose inner mapping group has order $2p$ (p is an odd prime), then* $\mathrm{Mlt}\, Q$ *is a solvable group. If in addition Q is finite, then Q is a solvable loop.*

Dihedral subgroup case

Our aim in this section is to prove the solvability of the multiplication group if the inner mapping group is a dihedral group of order $2p^n$, where p is an odd prime, and in case of finite loop in addition to show the solvability of the loop.

We prove our result by applying the technique of H-connected transversals. Our situation is the following: G is a group, H is a proper subgroup of G. There exist H and B H-connected transversals, i.e., A and B are left transversals to H such that $[A, B] \leq H$. We show that G is solvable provided that H is a dihedral group of order $2p^n$, where p is an odd prime number.

In our proof we use solvability criteria for finite groups and for factorized groups, some permutation group theory and finally a detailed analysis on how involutions are distributed among the cosets of H. We first show that the result is true for finite groups, and from this we easily get that it holds for infinite groups, too.

As we already mentioned in the beginning of this chapter, connected transversals are a very useful tool when studying the structure of loops. First we introduce some preliminary results on connected transversals and we prove our main theorem. Then we recall some basic facts and definitions about loops and we show how the group theoretical results established in this paper are applied to loop theory.

Connected transversals

In this section we give basic information about connected transversals and prove some preliminary lemmas, which are needed later. We assume that G is a group, $H \leq G$ and A, B are H-connected transversals in G. By $L_G(H)$ we denote the core of H in G (the largest normal subgroup of G contained in H).

Lemma 1.6.12 [CsMN, Lemma 2.3]**.** *If $C \subseteq A \cup B$ and $K = \langle H, C \rangle$, then $C \subseteq L_G(K)$.*

Now we prove three lemmas, where we assume that G is a group, $H \leq G$ is a dihedral group of order $2p^n$ (p is an odd prime number) and A and B are H-connected transversals in G. We denote by P the Sylow p-subgroup of H.

Lemma 1.6.13 [CsMN, Lemma 2.5]**.** *If H is a maximal subgroup of G and $L_G(H) = 1$, then $|H \cap H^g| \leq 2$ for every $g \in G \setminus H$.*

Proof. Clearly, $N_G(H) = H$ and thus $H \neq H^g$ whenever $g \in G \setminus H$. If $|H \cap H^g| > 2$, then $H \cap H^g$ contains a p-group $R \leq P$. Since H and H^g are dihedral, it follows that R is normal in $\langle H, H^g \rangle = G$, a contradiction. We conclude that $|H \cap H^g| \leq 2$. □

Lemma 1.6.14 [CsMN, Lemma 2.6]**.** *Let H be a maximal subgroup of G and $L_G(H) = 1$. If $aH = bH$, where $a \in A$ and $b \in B$, then $b^{-1}a = a^{-1}b \in H \cap H^{a^{-1}} = H \cap H^{b^{-1}}$.*

Proof. Now $a^{-1}b^{-1}ab \in H$, $Ha^{-1} = Hb^{-1}$ and $b \in aH$. Then $H^{a^{-1}} = H^{b^{-1}}$ and $b^{-1}a \in H \cap H^{a^{-1}}$. From Lemma 1.6.13 it follows that either $b^{-1}a$ is an involution or $b^{-1}a = 1$. In both cases $b^{-1}a = a^{-1}b$. □

Lemma 1.6.15 [CsMN, Lemma 2.7]**.** *Let G be a finite group, H a maximal subgroup of G and $L_G(H) = 1$. If $H \cap H^a = 1$ for some $1 \neq a \in A$, then $A = B$ and G is solvable.*

Proof. Let $H \cap H^a = 1$ and $aH = bH$, where $a \in A$ and $b \in B$. This means that $H \cap H^{a^{-1}} = 1$ and so $a = b$ by Lemma 1.6.14.

Assume that $uH = wH$, where $u \in A$ and $w \in B$. Now $a \in A \cap B$, hence $a^{-1}u^{-1}au \in H$. Thus $a^{-1}u^{-1}aw \in H$ and finally $a^{-1}u^{-1}wa \in H$. This means that $u^{-1}w \in H \cap H^{a^{-1}}$, hence $u = w$. We conclude that $A = B$.

Suppose that $b \in a^2H \cap A$. Now $a^{-1}b^{-1}ab \in H$, which means that $abH = baH$. Hence $a^{-1}baH = a^{-1}abH = a^2H$ and thus $b^a \in a^2H$. It follows that $(b^{-1}a^2)^a = (b^{-1})^a a^2 \in H$ and so $b^{-1}a^2 \in H^{a^{-1}}$. On the other hand $b^{-1}a^2 \in H$ and we get $b^{-1}a^2 \in H^{a^{-1}} \cap H = 1$. Thus $b = a^2$ and $a^2 \in A$. Now assume that $e \in A$, hence $e^{-1}e^{a^2}$ and $e^{-1}e^a$ are elements of H. Because $(e^{-1}e^a)^a H = (e^{-1})^a e^{a^2} H = (e^{-1})^a eH = H$, we conclude that $e^{-1}e^a \in H^{a^{-1}} \cap H = 1$. Thus $e \in C_G(a)$ and since e was an arbitrary element of A, we get that $A \subseteq C_G(a)$. Since $C_G(a) \cap H = 1$, we have $A = C_G(a)$.

Thus, A is a subgroup of G and since $[A, A] \leq A \cap H = 1$, it follows that A is an abelian group. Since $G = AH$ and $P \leq H$ is a nilpotent group, it follows from the Main Theorem in [Ca] that G is solvable. □

Main Theorem

We begin by introducing some general group theoretic results that are needed later in the proof of our main theorem.

Definition A nonidentity abelian subgroup T of a finite group G is said to be strongly self-centralizing if $C_G(t) = T$ for every nonidentity element t of T.

Lemma 1.6.16 [Co, pp. 124–126]**.** *Let T be a strongly self-centralizing subgroup of a finite group G and assume that $[N_G(T) : T] = 2$. If G is simple, then G contains exactly one conjugacy class of involutions.*

Lemma 1.6.17 [Co, pp. 99–103]**.** *Let G be a finite simple group having an elementary abelian Sylow 2-subgroup T of order 2^n ($n \geq 2$). If $C_G(t) = T$ for all nonidentity $t \in T$, then $G \cong SL(2, 2^n)$.*

Lemma 1.6.18 [Ve2, Theorem 4.2]**.** *Consider the projective special linear group $PSL(2, 2^n)$, where $n \geq 2$ and let H be a maximal subgroup of order $2(2^n + 1)$. Then there exist no H-connected transversals.*

After these preparations we are finally ready to prove

Theorem 1.6.19 [CsMN, Theorem 3.6]**.** *Let G be a group and let H be a dihedral subgroup of order $2p^n$, where p is an odd prime. If there exist H-connected transversals A and B in G, then G is solvable.*

Proof. First, let G be a finite group. We assume that G is a minimal counterexample.

Now H is dihedral and so every subgroup and factor group of H is dihedral or cyclic.

If $L_G(H) > 1$, then $H/L_G(H)$ is cyclic or dihedral and we have $H/L_G(H)$-connected transversals in $G/L_G(H)$. Hence $G/L_G(H)$ is solvable by induction or by Lemma 1.6.3. This means that also G is solvable and thus we may assume that $L_G(H) = 1$. Furthermore, $1 \in A \cap B$ by Lemma 1.6.1.

If H is not a maximal subgroup of G, then G has a proper subgroup K such that $H < K < G$. We may assume that $K = \langle H, C\rangle$, where $C \subset A$ and $C \neq 1$, and $L_G(K) > 1$ by Lemma 1.6.12. Now $HL_G(K)/L_G(K)$ is cyclic or dihedral and we have $HL_G(K)/L_G(K)$-connected transversals in $G/L_G(K)$. Hence $G/L_G(K)$ is solvable. Furthermore $A \cap K$ and $B \cap K$ are H-connected transversals in K and so K and also $L_G(K)$ and G are solvable. Thus, we may assume that H is a maximal subgroup of G. Furthermore, $N_G(H) = H$.

Next, we shall show that G is simple. If N is a nontrivial normal subgroup of G, then $G = NH$. We can write $H = PQ$, where $|P| = p^n$ and $|Q| = 2$. It is immediate that $N \cap H \leq P$. Let $g \in G \setminus H$, then $P^g \cap P = 1$, because $|H \cap H^g| \leq 2$ by Lemma 1.6.13. If we write $E = NP$, then E is a proper subgroup of G. Clearly, $N_E(P) = P$ and if $n \in E \setminus P$, then $P \cap P^n = 1$. Hence, E is a Frobenius group with Frobenius complement P. By using the properties of Frobenius groups [Hu1, p. 499], it follows that E is solvable and from this we conclude that $G = NH$ is solvable. Thus we may assume that G is simple.

If there exists $1 \neq a \in A$ such that $H \cap H^a = 1$, then G is solvable by Lemma 1.6.15. Thus we may assume that $H \cap H^a > 1$, whenever $1 \neq a \in A$. Because $N_G(P) = H$, P is a Sylow p-subgroup of G. Since $|P : P \cap P^g| = p^n$, whenever $g \in G \setminus H$, it follows that $[G : H] = 1 + kp^n$ [Hu1, p. 36]. If $1 + kp^n$ is an odd number, then Q is a Sylow 2-subgroup of G and Q has a normal complement. Because G is simple, we may assume that k is an odd number. We can consider G as a permutation group acting on the set with $1 + kp^n$ points and H is a one point stabilizer. Since $|H \cap H^g| = 2$, whenever $g \in G \setminus H$, we conclude that in the action of H on the remaining kp^n points, the orbits have length p^n. If we consider one orbit, it is clear that H acts transitively on the p^n points. By Lemma 1.6.4, every involution of H must fix one and only one point in the orbit. Thus, if $c \in H$ is an involution, c is a product of $k(p^n - 1)/2$ distinct transpositions. Thus, c and all its conjugates fix $k+1$ points. We note that if $g \in G \setminus H$ fixes a point, then $g \in H^x$ for some $x \in G \setminus H$. Again, we apply the counting argument from Lemma 1.6.4 and we obtain

$$|G| = 1 + kp^n + (1 + kp^n)(p^n - 1) + N(k + 1),$$

where N is the number of conjugates of c in G. From this result, we obtain that $N = (1 + kp^n)p^n/(k + 1)$. On the other hand, $N = [G : C_G(c)]$, which means that $|C_G(c)| = 2(k + 1)$. Since $k + 1 > 1$ and p does not divide $k + 1$ we conclude that $k + 1$ divides $1 + kp^n = (k + 1) + k(p^n - 1)$. Hence $k + 1$ divides $p^n - 1$.

We first study if it is possible that a coset aH, where $1 \neq a \in A$, contains more than two conjugates of c. Let c^x,c^y and c^z be three different conjugates of c. Now $c^x = ah$,$c^y = ak$ and $c^z = al$, where $h, k, l \in H$. This means that $c^x = (c^x)^{-1} = h^{-1}a^{-1}$ and $c^x c^y = h^{-1}k \in H$. Clearly, $c^x a^{-1}$ and $c^y a^{-1}$ are elements of $H^{a^{-1}}$ and thus $c^x c^y \in H \cap H^{a^{-1}}$. By Lemma 1.6.13 $c^x c^y$ is an involution and the three elements c^x, c^y and $c^x c^y$ commute with each other. Likewise $c^x c^z \in H$ is an involution, $c^x c^y \neq c^x c^z$ and $c^y c^z \in H$ is an involution, $c^y c^z \neq c^x c^y$ and $c^y c^z \neq c^x c^z$. Since $(c^x c^y)(c^x c^z)$ is a p-element from H, we have a contradiction. We conclude that every coset aH, $1 \neq a \in A$, has at most two conjugates of c.

The number of those conjugates of the involution c, which are not contained in H, is $N - p^n = kp^n(p^n - 1)/(k + 1)$. Every coset aH, where $1 \neq a \in A$, contains at most two conjugates of c, hence $kp^n(p^n - 1)/(k + 1) \leq 2kp^n$ and $p^n - 1 \leq 2(k + 1)$. Because $k + 1$ divides $p^n - 1$, we get $p^n - 1 = k + 1$ or $p^n - 1 = 2(k + 1)$. If $p^n - 1 = k + 1$, then $k = p^n - 2$ and $|G| = (1 + (p^n - 2)p^n)2p^n = (p^n - 1)^2 2p^n$. Let $|G| = 2^r s$, where s is odd. If R is a Sylow 2-subgroup of G, then $|R| = 2^r$. Since $Z(R) > 1$, it contains an involution z and $R \leq C_G(z)$. Now P is abelian and $C_G(x) = P$ for every $x \in P \setminus \{1\}$. This means that P is strongly self-centralizing and by Lemma 1.6.16, c and z are conjugate. It follows that 2^r divides $|C_G(c)|$ and so 2^{r-1} divides $k + 1$ and also $p^n - 1$. But then $(2^{r-1})^2 2 = 2^{2r-1}$ divides $|G|$, which means that $2^{2r-1} = 2^r$ or $r = 1$. Hence, $|G| = 2s$, where s is odd, which is not possible because the group G cannot have a Sylow 2-subgroup of order 2. Then we assume $2(k + 1) = p^n - 1$, and so the number of those conjugates of c, which are not contained in H, is $2kp^n$. Thus, every coset $aH \neq H$ has exactly two conjugates of c. Again all involutions of G are conjugate and we can assume that R is a Sylow 2-subgroup of G, which contains c and $c \in Z(R)$. Let $d = c^x$ and $f = c^y$ be two elements of aH, where $a \in C_G(c)H - H$ and $x, y \in G$. Now $c^d, c^f \in H$ and so $c \in H^d$ and $c \in H^f$. By Lemma 1.6.13, $c = c^d = c^f$, which means that $d, f \in C_G(c)$. Because $|C_G(c) \cap H| = 2$, we have $|C_G(c)H| = (p^n - 1)|H|/2$. We know that each coset $gH \neq H$ contains exactly two conjugates of c, thus we have at least $(p^n - 3) + 1$ involutions in $C_G(c)$. As $|C_G(c)| = p^n - 1$, we conclude that every element of $C_G(c) - \{1\}$ is an involution. Now $R \leq C_G(c)$ and we get that R is an elementary abelian Sylow 2-subgroup of order 2^r, where $r \geq 2$. Furthermore $p^n = 2^r + 1$ and $C_G(y) = R$ for every nonidentity $y \in R$. Hence we conclude by Lemma 1.6.17 that $G \cong SL(2, 2^r) = PSL(2, 2^r)$. Now $|H| = 2(2^r + 1)$, a contradiction by Lemma 1.6.18. We have proved that G is solvable in the case that G is a finite group.

Next we prove that our theorem also holds when G is infinite. Let $G = \langle A, B \rangle$ and first assume that $L_G(H) = 1$. Let a be a fixed element of A and h a fixed element of H and write $F(a, h) = \{b \in B : a^{-1}b^{-1}ab = h\}$. If b and c are elements of $F(a, h)$, then $bc^{-1} \in C_G(a)$ and $b \in C_G(a)c$. Thus $F(a, h) \subseteq C_G(a)b_h$, where b_h is a fixed element from $F(a, h)$, and $B = \bigcup F(a, h)$, where h goes through all the elements of H. Now $G = BH \subseteq C_G(a)\{b_h : h \in H\}H$ and thus $[G : C_G(a)] \leq |H|^2$ (likewise, $[G : C_G(b)] \leq |H|^2$ for any $b \in B$). Since $G = \langle A, B \rangle$ and H is finite, we conclude that $[G : C_G(H)]$ is finite, whence $[G : N_G(H)]$ is also finite. By Lemma 1.6.1, $N_G(H) = H \times Z(G)$ and thus $G/Z(G)$ is a finite group. By the first part of our proof $G/Z(G)$ is solvable and the solvability of G easily follows.

Then let $G = \langle A, B \rangle$ and assume that $L_G(H) > 1$. We write $\bar{G} = G/L_G(H)$ and $\bar{H} = H/L_G(H)$. If $\bar{H}$ is cyclic, then $\bar{G}$ is solvable by Lemma 1.6.3. If $\bar{H}$ is dihedral, we use the fact that $L_{\bar{G}}(\bar{H})$ is trivial and then proceed as in the previous part of the proof. It is immediate that $\bar{G}$ is solvable. But then it is clear that G is solvable, too.

Finally, let $K = \langle A, B \rangle$ be a proper subgroup of G. Now A and B are $K \cap H$-connected transversals in K and it easily follows that K is solvable. Since $[G : K]$ is finite, we conclude that $G/L_G(K)$ is a finite group. Now $HL_G(K)/L_G(K)$ is cyclic or dihedral and the solvability of $G/L_G(K)$ follows. Since $L_G(K)$ is solvable, it follows that G is solvable. The proof is complete. □

Loop theoretical result

In 1996 Vesanen [Ve1] showed that the solvability of $\mathrm{Mlt}\, Q$ implies the solvability of Q if Q is a finite loop.

Combining the above-mentioned result of Vesanen and Niemenmaa and Kepka's Theorem 1.1.1 concerning the connection between connected transversals and multiplication group with our Theorem 1.6.19 we obtain the following:

Theorem 1.6.20 [CsMN, Theorem 4.2]**.** *If Q is a loop whose inner mapping group is dihedral of order $2p^n$ (p an odd prime), then* $\operatorname{Mlt} Q$ *is a solvable group. If Q is finite, the Q is a solvable loop.*

pq **case**

In this section our aim is to prove the solvability of the multiplication group in case the order of the inner mapping group is pq, where p and q are odd primes, $p > q$ and $p = 2q^n + 1$.

We prove our result by applying the theory of connected transversals.

Thus we consider the following group theoretical situation:

G is a group with proper subgroup H, there exist A and B H-connected transversals, i.e., A and B are left transversals to H and $[A, B] \leq H$. $|H| = pq$, p, q are odd primes, $p > q$, $p = 2q^n + 1$. We give a proof for the solvability of G.

In our proof we analyse the structure of the normalizers of the Sylow q-subgroups of H and we show how the structure of G is related to these normalizers. We also use the properties of the transfer homomorphism and at one point of our proof we take advantage of the the Odd Order Theorem.

At the end of this section we list some of the general group theoretic results that are needed later. First we prove some technical lemmas about connected transversals in the case that $|H| = pq$, where $p > q$ are odd prime numbers. The next part contains the proof of our main theorem. As mentioned before, there is a link between connected transversals and loop theory. The last section of this paper is dedicated to loop theory: we discuss the relation between connected transversals and multiplication groups of loops and we also consider how solvable loops are recognized by their multiplication groups. Finally, we apply our main result to loop theory and we obtain a new solvability criterion for finite loops.

Our notation is standard. As usual, by $L_G(H)$ we denote the core of H in G (the largest normal subgroup of G contained in H).

In our proofs we also need the following more general group theoretic results.

Lemma 1.6.21. *A finite group $G = AH$ is not nonabelian simple if $Z(A) \neq 1$ and $[G : A] = p^n$ for some prime p.*

Lemma 1.6.22. *Let G be a finite group and let $G = AH$, where A is abelian and H is a subgroup such that $|H| = pq$ where $p > q$ are prime numbers. Then G is solvable.*

For the proofs, see [Go, p. 131] and [N6, Lemma 2.5].

Preliminaries

In [NK2] Kepka and Niemenmaa established the following solvability result.

Theorem 1.6.23 [CsN2, Theorem 2.1]**.** *If H is finite and abelian, then G is solvable.*

Now we introduce a series of lemmas which are later needed in the proof of our main theorem. We assume that $|H| = pq$, where $p > q$ are two odd prime numbers. We denote by P the unique p-subgroup of H.

Lemma 1.6.24 [CsN2, Lemma 2.2]**.** *If H is a maximal subgroup of G and $L_G(H) = 1$, then $|H \cap H^g|$ divides q for every $g \in G \setminus H$.*

Proof. Clearly, $N_G(H) = H$. Thus $H \neq H^g$, whenever $g \in G \setminus H$. If $|H \cap H^g| = p$, then $P \leq H \cap H^g$. Now P is a normal subgroup of H and H^g, hence P is normal in $\langle H, H^g \rangle = G$. Since $L_G(H) = 1$, this is not possible. Thus either $|H \cap H^g| = 1$ or $|H \cap H^g| = q$ for every $g \in G \setminus H$. $\square$

Lemma 1.6.25 [CsN2, Lemma 2.3]**.** *Let H be a maximal subgroup of G and $L_G(H) = 1$. If $a \in A$ and $b \in B$ are such that $aH = bH$, then $b^{-1}a \in H \cap H^{a^{-1}} = H \cap H^{b^{-1}}$.*

Proof. Now $a^{-1}b^{-1}ab \in H$ and $b \in aH$. Then $b^{-1}a \in H \cap H^{a^{-1}} = H \cap H^{b^{-1}}$. $\square$

Lemma 1.6.26 [CsN2, Lemma 2.4]**.** *Let G be a finite group, H a maximal subgroup of G and $L_G(H) = 1$. If $H \cap H^a = 1$ for some $1 \neq a \in A$, then $A = B$ and G is solvable.*

Proof. Let $H \cap H^a = 1$ and $aH = bH$, where $a \in A$ and $b \in B$. Clearly, $H \cap H^{a^{-1}} = 1$ and from Lemma 1.6.25 it follows that $a = b$.

Then let $uH = wH$ where $u \in A$ and $w \in B$. Now $a \in A \cap B$, hence $a^{-1}u^{-1}au \in H$. Thus $a^{-1}u^{-1}aw \in H$ and finally $a^{-1}u^{-1}wa \in H$. It follows that $u^{-1}w \in H \cap H^{a^{-1}} = 1$, hence $u = w$. We conclude that $A = B$.

Then let $b \in a^2H \cap A$. Now $a^{-1}baH = bH = a^2H$ and thus $b^a \in a^2H$. It follows that $(b^{-1}a^2)^a = (b^{-1})^a a^2 \in H$ and $b^{-1}a^2 \in H \cap H^{a^{-1}} = 1$. Thus $b = a^2$. Now assume that $e \in A$. Then $e^{-1}e^{a^2}$ and $e^{-1}e^a$ are elements of H and since $(e^{-1}e^a)^a = (e^{-1})^a e^{a^2}$, we conclude that $e^{-1}e^a \in H \cap H^{a^{-1}} = 1$. Thus $e \in C_G(a)$ and this shows that, in fact, $A \subseteq C_G(a)$. Since $C_G(a) \cap H = 1$, we have $A = C_G(a)$.

Thus A is a subgroup of G and since $[A, A] \leq A \cap H = 1$, it follows that A is an abelian group. Thus $G = AH$ and by Lemma 1.2, G is solvable. The proof is complete. $\square$

In the following four lemmas we assume that G is finite.

Lemma 1.6.27 [CsN2, Lemma 2.5]**.** *Let H be a maximal subgroup of G and $L_G(H) = 1$. If Q and R are two different q-subgroups of H, then $N_G(Q)H \cap N_G(R)H = H$. Further, if G is nonsolvable and $g \in G$, then there exists a q-subgroup Q of H such that $g \in N_G(Q)H$.*

Proof. Let $d \in N_G(Q)H \cap N_G(R)H$ and $d \notin H$. Then $Q^d \leq H \cap H^d$ and $R^d \leq H \cap H^d$. Thus $Q^d = R^d$, a contradiction. Then assume that G is nonsolvable and $g \in G \setminus H$. By Lemma 1.6.26, $H \cap H^g > 1$. Thus $H \cap H^g = Q^g$, where Q is a q -subgroup of H. Thus $Q^g = Q^h$, where $h \in H$, $gh^{-1} \in N_G(Q)$ and $g \in N_G(Q)H$. □

Lemma 1.6.28 [CsN2, Lemma 2.6]**.** *Let $1 \neq a \in A$, $b \in B$ and $aH = bH$. If H is maximal in G, $L_G(H) = 1$ and $a \in N_G(Q)H$, then $a^{-1}b \in Q$ (of course, here Q denotes a q-subgroup of H).*

Proof. Now $Q^a = H \cap H^a$ and therefore $a^{-1}b \in H \cap H^{a^{-1}} = Q$ by Lemma 1.6.25. □

Lemma 1.6.29 [CsN2, Lemma 2.7]**.** *Let H be a maximal subgroup of G and $L_G(H) = 1$. If Q is a q-subgroup of H and $N_G(Q) \cap A$ contains more than one element, then it follows that $N_G(Q)H \cap A \subseteq N_G(Q)$.*

Proof. Let $1 \neq a \in N_G(Q) \cap A$. If $b \in B$ and $aH = bH$, then $b \in N_G(Q)$ by Lemma 1.6.28. Let $c \in N_G(Q)H \cap A$. Then $c^b \in cH$ and $Q^{b^{-1}cb} = Q^{cb} \leq H$. As $Q^c \leq H \cap H^{b^{-1}} = Q$, we obtain $c \in N_G(Q)$. □

Remark. A similar result is naturally true for $N_G(Q) \cap B$.

Lemma 1.6.30 [CsN2, Lemma 2.8]**.** *Let H be maximal in G and $L_G(H) = 1$. If $1 \neq b \in B$, $1 \neq c \in B$, $c \neq b$ and $b^{-1}c \in N_G(Q)$ for a q-subgroup Q of H, then $b \in N_G(Q)$.*

Proof. Let $a \in N_G(Q)H \cap A$. Since $a^b \in aH$ and $a^c \in aH$, we obtain $Q^{b^{-1}ab} \leq H$ and $Q^{c^{-1}ac} \leq H$. As $Q^{b^{-1}} = Q^{c^{-1}} = H^{b^{-1}} \cap H^{c^{-1}}$ and $Q^{b^{-1}a} = Q^{c^{-1}a} = H^{b^{-1}} \cap H^{c^{-1}}$, we conclude that $a \in N_G(Q^{b^{-1}})$. As $a \in N_G(Q)H$, it follows from Lemma 1.6.27 that $b^{-1} \in N_G(Q)$ and the proof is complete. □

Remark. A similar result is true for the elements of A.

Main Theorem

We are now ready to prove our main theorem. In the proof we use the Odd Order Theorem by Feit and Thompson [FT]: If G is a finite group of odd order, then G is solvable.

Theorem 1.6.31 [CsN2, Theorem 3.1]**.** *Let G be a group, $H \leq G$ and $|H| = pq$, where $p > q$ are odd prime numbers such that $p = 2q^m + 1$. If there exist H-connected transversals A and B in G, then G is solvable.*

Remark. Since $p = 2q^m + 1$, it follows that either $q = 3$ or $q = 3v + 2$ and m is odd.

Proof. We first assume that G is finite and our proof is by induction on the order of G. As in the proof of Theorem 3.1 in [N6], we may assume that H is a maximal subgroup of G, $L_G(H) = 1$ and G is a simple group. From Theorem 1.6.23 we conclude that H is not abelian.

If there exists $1 \neq a \in A$ such that $H \cap H^a = 1$, then G is solvable by Lemma 1.6.26. Thus we may assume that $H \cap H^g > 1$ whenever $g \in G \setminus H$. Let P denote the unique p-subgroup of H. Now $N_G(P) = H$ and thus P is a Sylow p-subgroup of G, hence $[G : H] = 1 + kp$.

If Q is a q-subgroup of H, then $|N_G(Q)H| = np$, where $n = |N_G(Q)|$. By Lemma 1.6.27, $|G| = p(np - pq) + pq = (1 + kp)pq$. From this it follows that $n = q(k + 1)$.

If k is even, then $|G|$ is odd, hence solvable by the Odd Order Theorem. Thus we may assume that k is odd and this means that the order of $N_G(Q)$ is even and $N_G(Q)$ contains an involution. As n divides $|G|$, we obtain that $k + 1$ divides $(1 + kp)p$. Since $k+1 = p$ is not possible, it follows that $k+1$ divides $1+kp = 1-p+(k+1)p$ and thus $k+1$ divides $p-1 = 2q^m$. Of course, $N_G(Q)/C_G(Q)$ is cyclic and the order of this group divides $q - 1$.

We first assume that $k+1 = 2$. Then $|G| = (1+p)pq$ and Q is a Sylow q-subgroup of G. If $N_G(Q) = C_G(Q)$, then we can use Burnside's normal complement theorem (see [Go, p. 252) and thus Q has a normal complement in G, a contradiction. If $C_G(Q) = Q$, then G is a doubly transitive permutation group acting on $1+p$ points (the cosets of H) and every non-identity element fixes at most two points. Thus G is a Zassenhaus group of degree $p + 1$ and it follows that G is in fact the group $PSL(2, p)$ (for the details, see [HuB, p. 286]). Now Vesanen [Ve2] has shown that G does not have H-connected transversals.

Thus we may assume that $2q$ divides $k + 1$ and $2q^2$ divides $|N_G(Q)|$.

We now divide the proof in two parts: We first assume that $N_G(Q) \cap A$ contains more than one element for a q-subgroup Q of H. Now we denote $N_G(Q)$ by N and $C_G(Q)$ by C and it follows from Lemma 1.6.29 that $NH \cap A \subseteq N$ and $NH \cap B \subseteq N$. Clearly, $N = (A \cap N)Q = (B \cap N)Q$ and $C = (A \cap C)Q = (B \cap C)Q$. Here $A \cap C$ and $B \cap C$ are Q-connected transversals in C. Since Q is normal in C, we conclude that $C' \leq Q$. It is obvious that C is solvable and $C = KR$, where R is a Sylow q-subgroup of C and K is a Hall q'-subgroup of C. As K is characteristic in KQ, we conclude that $C = K \times R$.

We want to show that R is a Sylow q-subgroup of G. If R is not a Sylow q-subgroup of G, then there exists $g \in G - N$ such that $R^g = R$. Since $C^g = K^g \times R^g$ and $Q \leq R$, we conclude that $C^g = C$. If $g = bh$ ($b \in B, h \in H$) and $a \in A \cap C$, then $a^g = a^{bh} \in a^h H \subseteq N_G(Q^h)H$. By Lemma 1.6.27, we conclude that $h \in Q$ and thus $a^b \in C$.

If $|C|$ is even, then C contains an involution s. If $s \in A$, then $s^d \in sH$ for each $d \in B$. Thus $sd^{-1}sd \in H$ and we obtain $d^{-1}sds \in H^s \cap H = Q$. Thus $s^d \in sQ$ and

$s^d = s$. Then $B \subseteq C_G(s)$ and $G = C_G(s)P$. It follows from Lemma 1.6.21 that G is not simple. Thus we may assume that $s \notin A$ and we have $a \in A \cap C$ such that $a = sx$, where $1 \neq x \in Q$. Now $a^b \in C$, hence $a^b = ay$, where $y \in Q$. Furthermore, $a^q = s = (a^b)^q$, which means that s commutes with a and a^b. As $(a^b)^q = s = s^b$, we obtain $b \in C_G(s)$. Now $sa \in Q$ and $(sa)^b = say = xy \in Q$. But then $b \in N$, a contradiction.

Thus we may assume that $|C|$ is odd. Let $u \in N - C$ be an involution. If $u \notin A$, then we have $a \in A \cap N$ such that $a = ut$, where $1 \neq t \in Q$. But then a is an involution and we may conclude that $A \cap N$ contains an involution u. Now $u^b \in uQ$, hence $u^b \in \langle u \rangle C$. We write $F = \langle u \rangle C$. As $F = (A \cap F)Q = (B \cap F)Q$, we have $F' \leq Q$. On the other hand, $F^b = F = (A \cap F)^b Q^b = (B \cap F)^b Q^b$. From this we conclude that $F' \leq Q^b$, hence $F' \leq Q \cap Q^b = 1$. Thus F is abelian and $u \in C$, a contradiction.

We conclude that R is a Sylow q-subgroup of G (remember that $R > Q$). Let $1 \neq a \in R \cap A$. Assume that $a^g \in R$ and $g = bh$, where $b \in B$ and $h \in H$. Now $a^g = a^{bh} \in (aH)^h = a^h H \subseteq N_G(Q^h)H$. We conclude that $h \in Q$. It follows that $a^b \in R$, hence $a^b \in aQ$ and $a^b = ax$, where $x \in Q$. Thus $a^g = a^{bh} = (ax)^h = ax$. Now we consider the transfer homomorphism $f : G \to R/R'$ (for the properties of the transfer, see [Go], p. 245–251). By Theorem 7.3.3 of [Go], we conclude that $f(a) = (ax)^{[G:R]}R'$. Now $R' \leq Q$ and as $ax \notin Q$, we have $f(a) \neq R'$ and therefore $a \notin Ker(f)$. Thus $1 \neq Ker(f) < G$ and this contradicts the simplicity of G.

Then follows the second part of the proof. We denote by $Q_1, \ldots, Q_p$ the q-subgroups of H and we assume that $N_G(Q_i) \cap A = 1$ for every i. Naturally, the same assumption can be made for the sets $N_G(Q_i) \cap B$. We remind the reader that the order of $N_G(Q_i)$ is $(k+1)q$ and $2q$ divides $k+1$.

We write $A_i = A \cap N_G(Q_i)H$, $B_i = B \cap N_G(Q_i)H$ and $L_i = \langle A_i \rangle$. Let t and b be two elements from B_i which are different from 1. If $d \in t^{-1}H \cap A$, then $td \in H \cap H^{t^{-1}} = Q_i$. If $a \in A$ satisfies $aH = bH$, then $a^t \in aH$ and as $a^{-1}b \in Q_i$, we obtain $b^t \in bH$. Since $b^d \in bH$ and $b \in B_i$, we obtain $b^{t^{-1}} \in bH$. Further, $((b^{-1})^{t^{-1}}b)^t \in H \cap H^t$ which means that $b^{-1}b^t \in {Q_i}^t$. Likewise, $t^{-1}t^b \in {Q_i}^b$. Since $b^{-1}b^t = (t^{-1}t^b)^{-1} \in {Q_i}^t \cap {Q_i}^b = 1$, we conclude that $bt = tb$. (If ${Q_i}^t \cap {Q_i}^b \neq 1$, then $tb^{-1} \in N_G(Q_i)$ and $b^{-1}t \in N_G(Q_i)^b$. By Lemma 1.6.30, $b \in N_G({Q_i}^b)$, a contradiction.) Thus we may assume that the elements in the sets B_i commute with each other; of course, the same is true for the sets A_i.

Assume that $a \in A_i$ and $b \in B_i$ are two elements different from 1 and $a = b$. Then a commutes with every element in the sets A_i and B_i. If $dH = tH$ ($d \in A_i$, $t \in B_i$) and $d^{-1}t \neq 1$, then $d^{-1}t \in Q_i$ by Lemma 1.6.28 and $a \in C_G(Q_i)$, a contradiction.

Then let $a, d \in A_i$ and $b, t \in B_i$ with $aH = bH$ and $dH = tH$. Now we may assume that $a \neq b$ and as $|A_i| > |Q_i|$, we may also assume that $a^{-1}b = d^{-1}t$ or

$da^{-1} = tb^{-1}$. It follows that $da^{-1} \in C_G(a) \cap C_G(b)$. Since $a^{-1}b \in Q_i$, we obtain $da^{-1} \in C_G(Q_i)$, hence by Lemma 1.6.30, $a \in N_G(Q_i)$, a contradiction. Thus in what follows we may assume that $A = B$.

Now we write $A_i = \{a_1 = 1, a_2, \dots, a_l\}$ ($l = k+1$). Let $1 \neq a \in A_i$. Then $a^{-1}A_i \subseteq N_G({Q_i}^a)H$. We write $b_j = a^{-1}a_jH \cap A$ and as ${b_j}^a \in b_jH = a^{-1}a_jH$, we conclude that ${b_j}^{-1}a^{-1}a_j \in H \cap H^{a^{-1}} = Q_i$. Now we have $k \neq t$ such that ${b_k}^{-1}a^{-1}a_k = {b_t}^{-1}a^{-1}a_t$. It follows that $b_t{b_k}^{-1} = a_t{a_k}^{-1}$. We denote this element by x. Clearly, $1 \neq x \in \langle A_i\rangle \cap \langle A_f\rangle$, where $Q_f = {Q_i}^a$ and $Q_f \neq Q_i$.

Let $d \in A \cap xH$. If $a \in A_i$, then $d^a \in dH = xH$, hence $x^{-1}d \in H \cap H^{a^{-1}} = Q_i$. Similarly, if $b \in A_f$, then $x^{-1}d \in H \cap H^{b^{-1}} = Q_f$. It follows that $x = d \in A$. Since $x = {a_k}^{-1}a_t \in N_G({Q_i}^{a_k})H$, it is clear that $x \notin A_i$.

Then let $g \in L_i = \langle A_i\rangle$. We write $e \in A \cap gH$. If $a \in A_i$, then $e^a \in eH = gH$, hence $g^{-1}e \in H \cap H^{a^{-1}} = Q_i$. On the other hand, $e^x \in eH = gH$, hence $g^{-1}e \in H \cap H^{x^{-1}} = {Q_i}^{a_k} \neq Q_i$. It follows that $g = e \in A$. Thus we may conclude that $L_i \subseteq A$ for every i.

Let $1 \neq a \in A_i$ and $1 \neq b \in A_j$, where $i \neq j$. Now a^2 and b^2 are elements of A and $[a,b][b,a^2] \in H$, hence $[b,a]^a \in H$ and we obtain $[b,a] \in H \cap H^{a^{-1}} = Q_i$. Further, $[b,a][a,b^2] \in H$ and we get $[a,b] \in H \cap H^{b^{-1}} = Q_j$. Thus $[a,b] \in Q_i \cap Q_j = 1$, whence $ab = ba$. But this means that $\langle A\rangle$ is an abelian group and as $G = \langle A\rangle H$, we use Lemma 1.6.22 and it follows that G is solvable.

Then assume that G is infinite. If $L_G(H) > 1$, then $H/L_G(H)$ is cyclic and $G/L_G(H)$ is solvable by Theorem 1.6.23 and clearly G is solvable.

Thus we can assume that $L_G(H) = 1$. Let $G = \langle A, B\rangle$ and let a and h be fixed elements from A and H. We write $F(a,h) = \{b \in B : a^{-1}b^{-1}ab = h\}$. If $b, c \in F(a,h)$, then $bc^{-1} \in C_G(a)$ and $b \in C_G(a)c$. Thus $F(a,h) \subseteq C_G(a)b_h$, where b_h is a fixed element from $F(a,h)$.

Now $B = \cup_{h\in H} F(a,h)$ and $G = BH \subseteq C_G(a)\{b_h : h \in H\}H$.

It follows that $[G : C_G(a)] \leq |H|^2$. Thus $[G : C_G(H)]$ is finite and therefore $[G : N_G(H)]$ is finite. As $N_G(H) = H \times Z(G)$, we obtain that $[G : Z(G)]$ is finite. Since $|HZ(G)/Z(G)| = pq$ we conclude that $G/Z(G)$ is solvable by the first part of our proof. But this means that G is solvable.

Then let $K = \langle A, B\rangle$ be a proper subgroup of G. Now A and B are $K \cap H$-connected transversals in K and thus it is clear that K is solvable. Since $[G : K]$ is finite, we conclude that $[G : L_G(K)]$ is finite. Now $HL_G(K)/L_G(K)$ is cyclic or of order pq and this implies the solvability of $G/L_G(K)$. From the solvability of $L_G(K)$ it follows that G is solvable. The proof is complete. □

Loop theoretical result

By combining Vesanen's theorem and Niemenmaa and Kepka's result (Theorem 1.1.1) with our Theorem 1.6.31 we obtain:

Theorem 1.6.32 [CsN2, Theorem 4.3]**.** *If Q is a finite loop such that the inner mapping group* $\operatorname{Inn} Q$ *is of order pq, where p and q are odd prime numbers such that $p = 2q^m + 1$, then Q is a solvable loop.*

FINITE GROUPS

Basic notions and notations

G	is everywhere a finite group.
$\pi(G)$:	the set of prime divisors of the order of G.
$\Phi(G)$:	the Frattini subgroup of G is the intersection of all maximal subgroups of G.
$F(G)$:	the Fitting subgroup of G is the unique maximal nilpotent normal subgroup of G.
$F^*(G)$:	the generalized Fitting subgroup of G is the set of all elements x of G which induce an inner automorphism on every chief factor of G.
Definition:	H is a Hall subgroup of G, if $(\lvert H\rvert, \lvert G:H\rvert) = 1$.
Definition:	We say that H is a subnormal subgroup of G, if there exists $H = H_0 \lhd H_1 \lhd \ldots \lhd H_k = G$. We denote it by $H \lhd\lhd G$.
Definition:	G is a T-group if every its subnormal subgroup is normal in G.
Definition:	H is an S-quasinormal subgroup of G, if it permutes with every Sylow subgroup of G (i.e. $HQ = QH$ for all Sylow subgroups Q of G).
Definition:	G is a T^*-group, if every its subnormal subgroup is S-quasinormal in G.
Definition:	K is an $\mathcal{H}$-subgroup of G, if $N_G(K) \cap K^g \subseteq K$ for every $g \in G$.
Definition:	G is supersolvable if every its chief factor is cyclic.
Definition:	G is p-nilpotent if G has a normal p-complement N i.e. $N \lhd G,\ G = NP$, $N \cap P = 1$ with $P \in \mathrm{Syl}_p(G)$.
Definition:	A class $\mathcal{F}$ of groups is called a *formation* if $\mathcal{F}$ contains all homomorphic images of a group in $\mathcal{F}$ and if G/M and G/N are in $\mathcal{F}$, then $G/M \cap N$ is in $\mathcal{F}$ for normal subgroups M, N of G. A *formation* $\mathcal{F}$ is said to be *saturated*, if $G/\Phi(G) \in \mathcal{F}$ implies $G \in \mathcal{F}$.

2.1. The influence of minimal subgroups on the structure of finite groups

Minimal subgroups and S-quasinormality

Many authors have investigated the structure of a finite group G under the assumption that certain minimal subgroups of G are well situated in G. Ito [Hu1, p. 283] showed that a group of odd order is nilpotent provided that all minimal subgroups of G lie in the center of the group. Buckley [Bu] proved that a group G of odd order is supersolvable (i.e. its every chief factor is cyclic) if all minimal subgroups of G are normal.

Clearly if H is a normal subgroup of a group G, then $HK = KH$ for every subgroup K of G, i.e. H permutes with every subgroup of G. A subgroup of G is called S-quasinormal (or π-quasinormal) if it permutes with all Sylow subgroups of G. Thus S-quasinormality can be considered as a weak form of normality. The concept of S-quasinormality (π-quasinormality) was introduced by Kegel [Ke].

Then many authors studied the influence of S-quasinormality of some subgroups which ensures the supersolvability of the group. Yokoyama [Yo1, Yo2] and Laue [La] extended the results of Ito and Buckley, using formation theory to generalize to notion of centrality.

Recall that a class $\mathcal{F}$ of groups is called a *formation* if $\mathcal{F}$ contains all homomorphic images of a group in $\mathcal{F}$ and if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for normal subgroups M, N of G. A *formation* $\mathcal{F}$ is said to be *saturated*, if $G/\Phi(G) \in \mathcal{F}$ implies $G \in \mathcal{F}$ (see [Hu1, p. 696]). Throughout this chapter $\mathcal{U}$ will denote the class of supersolvable groups. Clearly $\mathcal{U}$ is a formation. Since a group G is supersolvable iff $G/\Phi(G)$ is supersolvable [Hu1, VI. p. 173] it follows $\mathcal{U}$ is a saturated formation.

M. Asaad, A. Ballester-Bolinches and M. C. Pedraza Aguilera in [AsBP] proved the following statement:

Let $\mathcal{F}$ be a saturated formation, containing the class $\mathcal{U}$ of supersolvable groups. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every subgroup of H of prime order or order 4 is S-quasinormal in G, then $G \in \mathcal{F}$.

It is natural to limit the hypotheses on minimal subgroups to a smaller subgroup. So with M. Asaad in [ACs1] we generalized the above-mentioned result in the following way:

Theorem 2.1.1 [ACs1, Theorem 1]**.** *Let $\mathcal{F}$ be a saturated formation containing the class* $\mathbf{U}$ *and let G be a group. Equivalent are:*

(a) *$G \in \mathcal{F}$.*

(b) *There is a normal solvable subgroup H in G such that $G/H \in \mathcal{F}$ and the subgroups of prime order or order 4 of the Fitting subgroup $F(H)$ are S-quasinormal in G.*

Immediate corollaries of this theorem are:

Corollary 2.1.2 [ACs1, Corollary 1, 2]**.** i) *Suppose that G is a group with a normal solvable subgroup H such that G/H is supersolvable. If every subgroup of $F(H)$ of prime order or order 4 is S-quasinormal in G, then G is supersolvable.*

ii) *If G is solvable and every subgroup of $F(G)$ of prime order or order 4 is S-quasinormal in G, then G is supersolvable.*

Corollary 2.1.3 (Laue [La])**.** *If G is solvable and every subgroup of $F(G)$ of prime order or order 4 is normal in G, then G is supersolvable.*

Preliminary results

Lemma 2.1.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathbf{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every subgroup of H of prime order or order 4 is S-quasinormal in G, then $G \in \mathcal{F}$.*

Proof. See [AsBP, Theorem 1]. In particular, if $\mathcal{F} = \mathbf{U}$, a result of Shaalan [Sha, Theorem 3.1] follows.

□

Lemma 2.1.5 [ACs1, Lemma 2]**.** *If P is an S-quasinormal p-subgroup of G for some prime p, then P is normalized by every p'-element a of G.*

Proof. P is subnormal in G (see [We, p. 24, Corollary 6.3]). It suffices to show the assertion for a q-element a of G where q is a prime $\neq p$. We have $a \in Q$ for some $Q \in \mathrm{Syl}_q(G)$. Since P is a subnormal and therefore normal Sylow p-subgroup of $PQ = QP$, the assertion follows.

□

Lemma 2.1.6 [ACS1, Lemma 3]**.** *Let P be a p-group with $p > 2$. If a p'-automorphism a of P centralizes $\Omega_1(P)$, then a is the identity on P.*

Proof. See [Go, p. 184, Theorem 3.20]. □

Lemma 2.1.7 [ACs1, Lemma 4]**.** *Let P be a p-subgroup of G, where $p > 2$. Suppose, all subgroups of P of order p are S-quasinormal in G. If a is a p'-element of $N_G(P) \setminus C_G(P)$, then a induces in P a fixed point free automorphism.*

Proof. Suppose that the action of a on P is not fixed point free. Then there is $c \in P$ of order p such that $c^a = c$. The subgroup $V = \langle c \rangle \Omega_1(Z(P)) \leq P$ is elementary abelian. Moreover, V and all subgroups of V are $\langle a \rangle$-invariant by Lemma 2.1.5. Therefore the linear transformation induced by a on V is scalar. It is necessarily the identity as a fixes c. By the same argument, for every $h \in \Omega_1(P)$ with $o(h) = p$, we have that a induces the identity on $\langle h\rangle\Omega_1(Z(P))$. So a centralizes $\Omega_1(P)$. Now $a \in C_G(P)$ by Lemma 2.1.6. $\square$

Proof of Theorem 2.1.1. With the aid of the preceding lemmas we can now prove our theorem.

(a) $\Longrightarrow$ (b) If $G \in \mathcal{F}$, then (b) is true with $H = 1$.

(b) $\Longrightarrow$ (a) Suppose that the result is false and let G be a counterexample of minimal order. Let p be the smallest prime dividing $|F(H)|$ and let $P \in \mathrm{Syl}_p(F(H))$. Clearly $P \trianglelefteq G$. Once we know that G/P inherits the hypothesis (b), we conclude $G/P \in \mathcal{F}$ by the minimality of $|G|$. Applying Lemma 2.1.4 we are done.

So we have to prove that (b) is inherited by G/P. Clearly H/P is a solvable normal subgroup of G/P with $(G/P)/(H/P) \cong G/H \in \mathcal{F}$. Let $L/P = F(H/P)$, we have to show that the subgroup of prime order or 4 of L/P are S-quasinormal in G/P. Let X/P be a subgroup of L/P of prime order q (of order 2 or 4 if $q = 2$). Let Q be a Sylow q-subgroup of L. We have:

i) $p \nmid |L/P|$, in particular, $q \neq p$:
If P_1/P is the Sylow p-subgroup of the nilpotent characteristic subgroup L/P of H/P, then $P_1 \trianglelefteq G$ and thus $P_1 \leq F(H)$, i.e. $P_1 = P$.

ii) Q is not cyclic:
Otherwise the Sylow q-subgroup QP/P of L/P is cyclic. Since $X/P \leq QP/P$, then X/P is a characteristic subgroup of QP/P, it follows $X/P \trianglelefteq G/P$ which certainly implies that X/P is S-quasinormal in G/P.

iii) L is supersolvable:
This follows by Lemma 2.1.4 since L/P is nilpotent, in particular supersolvable, and the subgroups of order p (of order 2 or 4 if $p = 2$) of P are S-quasinormal in L.

iv) $q < p$, in particular, $p > 2$ and $q \nmid |F(H)|$:
As QP/P is the Sylow q-subgroup of L/P, we have that $QP \trianglelefteq L$ and QP is supersolvable by iii). If $q > p$, then $Q \trianglelefteq QP$ [We, p. 6, Theorem 1.8], which means that $QP = Q \times P \leq F(H)$. Thus, $X \cap Q$ is S-quasinormal in G by hypothesis and since $X = P(X \cap Q)$ it follows that X/P is S-quasinormal in G/P. So $q < p$, $p > 2$, and $q \nmid |F(H)|$ by choice of p.

v) Q acts faithfully and (elementwise) fixed point freely on P:
Let $F(H) = P \times T$. So $QT \cong QP/P \times TP/P \leq L/P$ is nilpotent. Therefore Q centralizes T. If some $a \in Q$ centralizes also P, then $a \in C_H(F(H)) \cap Q \leq F(H) \cap Q = 1$, as H is solvable. So $a = 1$. Now, by Lemma 2.1.7, every nonidentity element of Q induces a fixed point free automorphism on P.

vi) Finish of the proof:
By v) and [Go, p. 339, Theorem 3.1], Q is cyclic or generalized quaternion. Since L is supersolvable, one knows $L' \leq F(L) = F(H)$. So $L/F(H)$ is abelian and since $q \nmid |F(H)|$, it follows that $Q \cong QF(H)/F(H)$ is abelian, that is cyclic. This is in contradiction with item ii). □

A further consequence of Theorem 2.1.1:

Corollary 2.1.8 [ACs1, Corollary 4]**.** *Let $\mathcal{F}$ be a saturated formation containing* $\mathbf{U}$*. Suppose that G is a group with a normal solvable subgroup H such that $G/H \in \mathcal{F}$. Then $G \in \mathcal{F}$ under either of the following conditions:*

a) *G is 2-nilpotent and every subgroup of odd prime order of $F(H)$ is S-quasinormal in G.*

b) *The Sylow 2-subgroups of G are abelian and every subgroup of $F(H)$ of prime order is S-quasinormal in G.*

Proof. a) Suppose G is 2-nilpotent and let K be the normal 2-complement of G. Then $K_1 = K \cap H$ is the 2-complement of H. Since H/K_1 is G-isomorphic with HK/K, we see that H/K_1 is in the hypercenter of G/K_1. It follows that also $G/K_1 \in \mathcal{F}$ by the hypothesis on $\mathcal{F}$. Also $F(K_1) \leq F(H)$. So we may substitute H by K_1 which is of odd order. The result now follows applying Theorem 2.1.1.

b) Let P be an abelian Sylow 2-subgroup of G and put $P_1 = F(H) \cap P$. It follows that P_1 is in the center of G, since P is abelian and, by hypothesis, every Sylow subgroup of odd order of G centralizes $\Omega_1(P_1)$ and P_1 [Go, p. 178, Theorem 2.4]. The rest of the assertion now follows by the proof of our theorem since G/P_1 inherits the hypothesis. □

Remarks. (a) The theorem is not true for saturated formations, which do not contain $\mathbf{U}$. For example, if $\mathcal{F}$ is the saturated formation of all nilpotent groups, then the symmetric group of degree three is a counterexample.

(b) The theorem is not true if we omit the solvability of H. Set $G = H \times K$, where $H = \mathrm{SL}(2,5)$ and $K \in \mathbf{U}$. Then $|F(H)| = 2$ and $G/H \cong K \in \mathbf{U}$, but $G \notin \mathbf{U}$.

(c) Let $G = S_3 \times C$, where S_3 is the symmetric group of degree three and C a cyclic group of order 3. Clearly, G is supersolvable. It is easy to check that G contains a subgroup of order three fails to be S-quasinormal in G, and so the converse of Corollary 2.1.2 is false.

A corollary of Corollary 2.1.8 is the following result: If G is a solvable group and every subgroup of the Fitting subgroup $F(G)$ of prime order or order 4 is S-quasinormal in G, then G is supersolvable. Examining the structure of this subclass of supersolvable groups we obtain the following result:

Theorem 2.1.9 [Cs1, Theorem 5]**.** *Let G be a solvable group. Then every subgroup of $F(G)$ of prime order or order 4 is S-quasinormal in G if and only if $G = M(N \times K)$, where M is a nilpotent normal subgroup of odd order, N is a nilpotent subgroup, K is a nilpotent Hall subgroup such that $M \cap (N \times K) = 1$, $F(G) \cap N = 1$ and every minimal subgroup of M is normal in M and S-quasinormal in G.*

For the proof we need the following results.

Lemma 2.1.9 (1) [Cs1, Lemma 3]**.** *Let U be a 2-group in a group G, $a \in N_G(U)$ with $(o(a), 2) = 1$ and a normalizes every minimal subgroup of U and every cyclic subgroup of order 4. Then $a \in C_G(U)$.*

Proof. Let h be an element of U of order 4. By the conditions either $h^a = h$ or $h^a = h^3$. If $h^a = h^3$, then $a^2 \in C_G(h)$. Since a is of odd order, we conclude $a \in C_G(h)$. Consequently $a \in C_G(\Omega_2(U))$, which yields $a \in C_G(U)$. $\square$

Lemma 2.1.9 (2) [Cs1, Lemma 4]**.** *Let P be a normal p-subgroup of a solvable group G with an odd prime p. Suppose every minimal subgroup of P is S-quasinormal in G. Then one of the following holds:*

(1) *every minimal subgroup of P is normal in P.*

(2) *$Q \leq C_G(P)$ for every Sylow q-subgroup Q of G with $q \neq p$.*

Proof. Assume there exists an element x_0 of P such that $o(x_0) = p$ and $\langle x_0 \rangle$ is not normal in P. The solvability of G implies the existence of a Hall subgroup H with $\pi(H) = \pi(G) \setminus \{p\}$. As P is normal in G, $\langle x_0 \rangle$ is subnormal in G. From the S-quasinormality of $\langle x_0 \rangle$ we easily conclude that $H \leq N_G(\langle x_0 \rangle)$. Let H_1 be the normal closure of H in G. Obviously $H_1 \leq N_G(\langle x_0 \rangle)$. As $\langle x_0 \rangle$ is not normal in G, we find that $H_1 \cap P = P_0 \neq P$. Using $P \triangleleft H_1P$ and $H_1 \triangleleft H_1P$ we have that the elements of H fix the elements of P/P_0 by conjugation. Applying Glauberman's Theorem [Gla] we get that there exists $\nu \in P \setminus P_0$ such that $H \leq C_G(\nu)$. Clearly the elements of H normalize every minimal subgroup of P. Applying Lemma 2.1.7 $H \leq C_G(P)$ holds. Let Q be a Sylow q-subgroup of G with $q \neq p$. As $Q \leq H^z$ for some $z \in G$ and P is normal in G, our statement follows. $\square$

Proof of Theorem 2.1.9. Suppose every subgroup of $F(G)$ of order prime or 4 is S-quasinormal in G. It follows from Corollary 2.1.8 that G is supersolvable, whence G' is nilpotent and $G' \leq F(G)$. Using [Hu2, Satz 3.10 p. 271] $G = HF(G)$ for some nilpotent subgroup H. If P is an arbitrary Sylow subgroup of $F(G)$, denote by P^* the unique Sylow subgroup of the subgroup HP containing P. Denote by $\mathcal{S}$ the set of those Sylow subgroups P of $F(G)$ for which HP is nilpotent. Define $\mathcal{S}^* = \{P^* \mid P \in \mathcal{S}\}$. Suppose ${P_1}^*, {P_2}^* \in \mathcal{S}^*$. From the above we can easily conclude that ${P_1}^* \leq C_G({P_2}^*)$. Let K be the direct product of the elements of $\mathcal{S}^*$. We have

$F(G) = M \times (K \cap F(G))$ and $H = N \times (H \cap K)$, for some nilpotent subgroup N and a nilpotent Hall subgroup M. Let B be the Sylow 2-subgroup of $F(G)$ and h an arbitrary element of odd order of H. By the conditions, using Lemma 2.1.9(1), $h \in C_G(B)$ holds, whence BH is nilpotent; consequently M is of odd order. We have $NK = N \times K$, $M \triangleleft G$, $NM \triangleleft G$, and so we find $G = M(N \times K)$. Assume Q is a Sylow subgroup for an odd prime of $F(G)$ with $H \cap Q \neq 1$. Obviously $H = (H \cap Q^*) \times T$ for some Hall subgroup T of H. Hence $T \leq C_G(H \cap Q)$ and $T \leq N_G(Q)$. By using our hypothesis and applying Lemma 2.1.7 we get $T \leq C_G(Q)$. Thus in this case $Q^* \in \mathcal{S}^*$, that is $Q^* \leq K$. This fact implies $M \cap N = 1$ and $M \cap (N \times K) = 1$. Let R be an arbitrary Sylow subgroup of M. By our Lemma 2.1.9(2) every minimal subgroup of R is normal in R and consequently in M too, and is S-quasinormal in G. Obviously $F(G) \cap N = 1$.

Assume conversely $G = M(N \times K)$ has the required properties. Clearly $M \leq F(G) \leq MK$. Let D be a subgroup of $F(G)$ of prime order or order 4. Supposing $D \leq M$, D is S-quasinormal in G. If $D \leq K$, using the structure of G, it is easy to see the S-quasinormality of D in G. □

For the description of the structure of Sylow subgroups of M in the previous theorem we can apply the following theorem, so we get a complete characterization:

Theorem 2.1.10 [Cs1, Theorem 4]**.** *Let G be a supersolvable group and P is a normal p-subgroup of G with $p \neq 2$. Then every minimal subgroup of P is normal in P and S-quasinormal in G if and only if there is a chain $1 = P_0 \triangleleft P_1 \triangleleft \ldots \triangleleft P_k = P$ with $P_i \triangleleft G$, $|P_i/P_{i-1}| = p$ for every $1 \leq i \leq k$ and $\Omega_1(P) = P_\ell \leq Z(P)$, for some $1 \leq \ell \leq k$. Moreover, for every $g \in G$ with $(o(g), p) = 1$, there exists a natural number t_g with $1 \leq t_g \leq p-1$ such that $a^g = a^{t_g}$, where a is an arbitrary element of $D = \bigtimes_{i=1}^{k} (P_i/P_{i-1})$.*

For the proof of this latter Theorem 2.1.10 we need our natural factorization of supersolvable groups, so the proof can be founded in Chapter 2.2.

It is meaningful to remove the solvability of H in the hypotheses of Theorem 2.1.1, but in this case the Fitting subgroup $F(H)$ will sometimes be a trivial group, while the generalized Fitting subgroup $F^*(G)$ is never trivial if $G \neq 1$ (see [HuB, X. 13]).

Recall that for any group G, the generalized Fitting subgroup $F^*(G)$ is the set of all elements x of G which induce an inner automorphism on every chief factor of G. Clearly $F^*(G)$ is a characteristic subgroup of G. By [Hu2, III. 4.3] $F(G) \leq F^*(G)$.

In [LW1, Theorem 3.1] Li and Wang extended our result obtaining the following: Suppose that G is a group with a normal subgroup N such that G/N is supersolvable.

If every subgroup of prime order or order 4 of $F^*(N)$ is S-quasinormal in G, then G is supersolvable. Their proof depends heavily upon our proof.

Later with M. Asaad in [ACs2] we gave a characterization of a group G under the assumption that every subgroup of $F^*(G)$ of prime order or order 4 is S-quasinormal in G.

More precisely, we proved the following:

Theorem 2.1.11 [ACs2, Theorem 1.1]**.** *Let G be a group of composite order such that the quaternion group of order 8 is not involved in G.*

Then the following statements are equivalent:

(1) *Every subgroup of $F^*(G)$ of prime order is S-quasinormal in G.*

(2) *$G = UW$, where U is a normal nilpotent Hall subgroup of odd order and W is a supersolvable Hall subgroup and $(|U|, |W|) = 1$ and every subgroup of U of prime order is S-quasinormal in G.*

(3) *G is solvable and every subgroup of $F(G)$ of prime order is S-quasinormal in G.*

For the proof of Theorem 2.1.11 we need the following:

Lemma 2.1.12 [ACs2, Lemma 2.1]**.** *Let $S \in \mathrm{Syl}_2(G)$. If every subgroup of S of order 2 is normal in G and the quaternion group of order 8 is not involved in G, then G has a normal 2-complement.*

Proof. Let G be a counterexample of minimal order. Then G has not a normal 2-complement, so G contains a minimal non-2-nilpotent subgroup K. By [Hu2, IV. Satz 5.4] K is a minimal non-nilpotent subgroup of G. By [Hu2, III. Satz 5.2], $|K| = 2^n q^m$ for a prime $q \neq 2$, K has a normal Sylow 2-subgroup K_2 and a cyclic Sylow q-subgroup K_q, the exponent of K_2 is 2 or 4. Hence if the exponent of K_2 is 2, then K_2 is elementary abelian and so $K_q \trianglelefteq K$, a contradiction. Thus the exponent of K_2 is 4. By [Hu2, III. Satz 5.2] $K_2' = Z(K_2) = \Phi(K_2) \leq Z(K)$. Let R be a maximal subgroup of K_2'. Then $R \trianglelefteq K$ and so K/R is a minimal non-nilpotent group and $K_2'/R = (K_2/R)' = Z(K_2/R) = \Phi(K_2/R)$ and $|K_2'/R| = 2$, so K_2/R is an extraspecial quaternion free 2-group. Then, by [Go, Chap. 5, Theorem 5.2] K_2/R is isomorphic to a dihedral group of order 8, so K_2/R contains a characteristic subgroup of order 4. This implies that K/R is nilpotent, a final contradiction. $\square$

Lemma 2.1.13 [ACs2, Lemma 2.2]**.** *If every subgroup of prime order of $F^*(G)$ is normal in G and the quaternion group of order 8 is not involved in G, then G is supersolvable.*

Proof. If $F^*(G)$ is of odd order, then G is supersolvable by [La]. Thus we may assume that $F^*(G)$ is of even order. By [Hu2, III. Satz 5.3b] $F^*(G)$ is solvable.

Hence using Lemma 2.1.12 we get $F^*(G) = F(G)$. Let $S \in \mathrm{Syl}_2(F(G))$. Then $S \trianglelefteq G$ and SQ is a subgroup of G for every $Q \in \mathrm{Syl}(G)$ with $(|Q|, 2) = 1$. By Lemma 2.1.12 $SQ = S \times Q$ and since $S \trianglelefteq G$, it follows easily that every subgroup of order 4 of S is S-quasinormal in G. Therefore every subgroup of prime order or order 4 of $F^*(G)$ is S-quasinormal in G. Let H be any subgroup of order 4 of S. Then $O^2(G) \leq N_G(H) \leq G$. Hence if $N_G(H) = G$ for all $H \leq S$, $|H| = 4$, we have every subgroup of prime order or order 4 of $F^*(G) = F(G)$ is normal in G. Using [La] we get G is supersolvable. Thus there exists $H \leq S$ such that $|H| = 4$ and $N_G(H) < G$. But $O^2(G) \leq N_G(H) < G$. Since $F^*(O^2(G)) \leq F^*(G)$, by [HuB, X. 13, 11 Corollary] we have $O^2(G)$ is supersolvable by induction. As $G/O^2(G)$ is a 2-group, we have G is solvable. Now applying Theorem 2.1.1 G is supersolvable. □

Proof of Theorem 2.1.11. (1) $\Longrightarrow$ (3) By induction on the order of G. First we argue that G is solvable. If every minimal subgroup of $F^*(G)$ is normal in G, then G is supersolvable by Lemma 2.1.13, and so G is solvable. Thus we assume that there exists a subgroup H of prime order p of $F^*(G)$ such that H is not normal in G. By hypothesis H is S-quasinormal in G and so $O^p(G) \leq N_G(H) < G$. But $F^*(O^p(G)) \leq F^*(G)$ by [HuB, X. 13. 11 Corollary] and since every subgroup of $F^*(O^p(G))$ of prime order is S-quasinormal in $O^p(G)$, then by induction on $|G|$, $O^p(G)$ is solvable and since $G/O^p(G)$ is a p-group, we have G is solvable.

Now by [HuB, X. 13], $F^*(G) = F(G)$, so every subgroup of $F(G)$ of prime order is S-quasinormal in G.

(3) $\Longrightarrow$ (2) We argue that every subgroup of $F(G)$ of order 4 is S-quasinormal in G. Let $S \in \mathrm{Syl}_2(F(G))$. Let $\overline{M} = M/F(G)$ be a $2'$-Hall subgroup of $\overline{G} = G/F(G)$. Thus $|\overline{G} : \overline{M}| = 2^t$. Hence if $\overline{M} = 1$, $|\overline{G}| = |G/F(G)| = 2^t$ and so it follows easily that every subgroup of $F(G)$ of order 4 is S-quasinormal in G.

Thus we may assume that $\overline{M} \neq 1$. Consider $O^2(M) = L$. We distinguish the following two cases:

Case 1. $L \cap S \neq 1$. Clearly $L \trianglelefteq M$ and every subgroup of $L \cap S$ of order 2 is S-quasinormal in L and so easily every subgroup of $L \cap S$ of order 2 is normal in L. Obviously $L \cap S \in \mathrm{Syl}_2(L)$, consequently L has a normal 2-complement, i.e. $L = (L \cap S) \times T$. Since $T \operatorname{char} L \trianglelefteq M$ then $T \trianglelefteq M$, using $S \trianglelefteq M$ we get $ST = S \times T$. Set $Q \in \mathrm{Syl}_q(G)$ with $q \neq 2$, let $S_0 \leq S$ be such that $|S_0| = 4$. Clearly $Q^y \leq T$ for some $y \in G$, whence ${S_0}^y \leq C_G(Q^y)$, and we can conclude $S_0 Q$ is a subgroup of G.

Case 2. $L \cap S = 1$. Then $M = L \times S$. Let $Q \in \mathrm{Syl}_q(G)$ with $q \neq 2$. Clearly $Q^z \leq M$ for some $z \in G$ and so $Q^z \leq L$. Let $S_0 \leq S$ be such that $|S_0| = 4$. Clearly ${S_0}^z \leq S$, whence it follows $Q^z \leq C_G({S_0}^z)$, consequently $Q \leq C_G(S_0)$.

Thus G is solvable and every subgroup of $F(G)$ of prime order or of order 4 is S-quasinormal in G. Then applying Corollary 2.1.8 we get the supersolvability of G. Let p be the largest prime dividing $|G|$ and $P \in \mathrm{Syl}_p(G)$, the supersolvability of G implies $P \trianglelefteq G$. Now let U be a normal nilpotent Hall subgroup of G of odd order such that $P \leq U$. Then $G = UW$, where W is a subgroup of G such that $(|U|, |W|) = 1$. Clearly $U \leq F(G)$ and hence every subgroup of U of prime order is S-quasinormal in G.

(2) $\Longrightarrow$ (3) By induction on the order of G. G is supersolvable by [Sha, Theorem 3.1] and we can conclude $G' \leq F(G)$. Let M be a maximal subgroup of G containing $F(G)$. Then $M \triangleleft G$ so $F(M) = F(G)$. Also $M = U(M \cap W)$ and so by induction on $|G|$ we have every subgroup of $F(M)$ $(= F(G))$ of prime order is S-quasinormal in M. Now we argue that every subgroup of $F(G)$ of prime order is S-quasinormal in G. Let $H \leq F(G)$, where H is of prime order. Since $F(G) = U \times (F(G) \cap W)$ we have either $H \leq U$ or $H \leq F(G) \cap W$. If $H \leq U$ then H is S-quasinormal in G by hypothesis. Assume $H \leq F(G) \cap W$, write $|G/M| = r$, where r is a prime number. Clearly, H permutes with every Sylow subgroup Q of G such that $(|Q|, r) = 1$. Now let $R \in \mathrm{Syl}_r(G)$. We claim that HR is a subgroup of G. If $|H| = r$, then let $R^* \in \mathrm{Syl}_r(F(G))$. Clearly $R^* \trianglelefteq G$, consequently $R^* \leq R$. Since $H \leq R^* \leq R$ then $HR = RH = R$. If $|H| \neq r$, let $|H| = q \neq r$ and $Q^* \in \mathrm{Syl}_q(F(G))$. Then $Q^* \trianglelefteq G$ and Q^*R is a subgroup of G. Hence if $R \trianglelefteq Q^*R$, then $Q^*R = Q^* \times R$, and we can conclude HR is a subgroup of G. Thus we may assume that R is not normal in Q^*R. But Q^*R is supersolvable, so $q > r$. If $Q^*R \lneqq G$, then by the induction on $|G|$ we have that every subgroup of prime order of $F(Q^*R)$ is S-quasinormal in Q^*R and since $H \leq Q^* \leq F(Q^*R)$ it follows HR is a subgroup of G. If $Q^*R = G$, then $Q^* = U$, whence $H \leq U$, which is a contradiction with $H \leq F(G) \cap W$ and $(|U|, |F(G) \cap W|) = 1$.

(3) $\Longrightarrow$ (1) We have that G is solvable, hence $F^*(G) = F(G)$ by [HuB, X. 13], consequently every subgroup of $F^*(G)$ of prime order is S-quasinormal in G. $\square$

Later Y. Li and Y. Wang in [LW2] extended our result Theorem 2.1.1 replacing S-quasinormality by requiring that the subgroups of prime order or order 4 of the generalized Fitting subgroup of some normal subgroup are S-quasinormally embedded in the whole group. We recall a subgroup H of a group G is said to be S-quasinormally embedded (or π-quasinormally embedded) in G, if for each prime p dividing the order of H a Sylow p-subgroup of H is also a Sylow p-subgroup of some S-quasinormal subgroup of G. The proof of Lie and Wang's theorem is strongly relied on our result.

Recently Asaad and Heliel [AsH] generalized the notion of S-quasinormality, introducing a weaker new embedding property, namely the 3-permutability (or Σ-permutability) of a subgroup of a group. 3 is a complete set of Sylow subgroups of the group G, if for each prime divisor p of the order of G 3 contains exactly one Sylow p-subgroup of G. A subgroup of G is said to be 3-permutable if it permutes with every member of 3. Obviously every S-quasinormal subgroup is 3-permutable. Heliel, Xianghua Li and Yangming Li in [HLL] extended our Theorem 2.1.1 replacing the S-quasinormality by 3-permutability. In the proof of their result our Theorem 2.1.1 was used very heavily. Beside our result they needed the classification of finite simple groups. Three years later Li Fang Wang and Yan Ming Wang [WW] gave an elementary proof for the previous statement without using the classification.

Minimal subgroups and $\mathcal{H}$-property

In this section we wish to present some new sufficient condition for supersolvability which involve that certain subgroups of a group G possess the $\mathcal{H}$-property.

Following Bianchi et al. [BMHV] a subgroup K of a group G is called an $\mathcal{H}$-subgroup of G if the following condition is satisfied

$$N_G(K) \cap K^g \subseteq K \quad \text{for every} \quad g \in G.$$

Denote by $\mathcal{S}p(G)$ the set of all cyclic subgroups of G of prime order or of order 4 and by $\mathcal{H}(G)$ the set of all $\mathcal{H}$-subgroups of G. We propose to call the supersolvable residual of G *the* $\mathcal{S}p$ *supersolvator of* G (i.e. that minimal normal subgroup, the factor over it is supersolvable) and denote it by $\operatorname{sups} G$. Denote by $\mathcal{N}(G)$ the set of all normal subgroups of G, and by $\mathcal{S}u(G)$ the set of all subgroups of G. We prove that the condition $\mathcal{S}p(G) \subseteq \mathcal{H}(G)$ implies the supersolvability of G. Then we characterize structurally all supersolvable groups satisfying that condition. We also show that if $H \trianglelefteq G$ and G/H is supersolvable, then the condition $\mathcal{S}p(H) \subseteq \mathcal{H}(G)$ implies the supersolvability of G. If H is also solvable, then the conclusion holds under the weaker assumption $\mathcal{S}p(F(H)) \subseteq \mathcal{H}(G)$ ($F(H)$ is the Fitting subgroup of H). As a corollary we obtain the result that if $\mathcal{S}p(\operatorname{sups} G)) \subseteq \mathcal{H}(G)$, then G is supersolvable and if $\operatorname{sups}(G)$ is solvable, then it suffices to require $\mathcal{S}p(F(\operatorname{sups} G)) \subseteq \mathcal{H}(G)$.

First we list the properties of $\mathcal{H}$-subgroups of a group G which were proved by Bianchi et al. in [BMHV] and which will be frequently used.

Lemma 2.1.14 [CsH, Lemma 3]**.**

(1) *If $N \leq H \leq G$ and $N \trianglelefteq G$, then $H \in \mathcal{H}(G)$ if and only if $H/N \in \mathcal{H}(G/N)$.*

(2) *If $H \trianglelefteq\trianglelefteq K \leq G$ and $H \in \mathcal{H}(G)$, then $H \trianglelefteq K$.*

(3) *If $H \leq K \leq G$ and $H \in \mathcal{H}(G)$, then $H \in \mathcal{H}(K)$.*

(4) *If $K \leq G$, then $\mathcal{H}(G) \cap \mathcal{S}u(K) \subseteq \mathcal{H}(K)$.*

Proof. For (1), see Bianchi et al. [BMHV, Lemma 2(1)]; for (2) [BMHV, Theorem 6(2)], for (3), see [BMHV, Lemma 7(2)]. Finally, (4) follows immediately from (3). $\square$

In addition to these results we need the following two lemmas.

Lemma 2.1.15 [CsH, Lemma 5]**.** *Let M and L be subgroups of G and suppose that $LM = ML$, $(|L|, |M|) = 1$ and $G = MN_G(L)$. Then $L \in \mathcal{H}(G)$.*

Proof. Let $g \in G$; then $g = nm$ with $n \in N_G(L)$ and $m \in M$. Thus $L^g = L^{nm} = L^m \leq LM$ and

$$L^g \cap N_G(L) \leq LM \cap N_G(L) = L(N_G(L) \cap M).$$

Since $(|L|,|M|)=1$ and $L \trianglelefteq L(N_G(L)\cap M)$, it follows that $L^g \cap N_G(L) \leq L$, whence $L \in \mathcal{H}(G)$. $\square$

Lemma 2.1.16 [CsH, Lemma 6]**.** *Suppose that $N \triangleleft G$, Q is a p-subgroup of G which belongs to $\mathcal{H}(G)$ and $(|N|,|Q|)=1$. Then $QN \in \mathcal{H}(G)$ and $(QN)/N \in \mathcal{H}(G/N)$.*

Proof. By Lemma 2.1.14(1) it suffices to prove that $QN \in \mathcal{H}(G)$, i.e.,

$$N_G(QN) \cap (QN)^x \leq QN \quad \forall x \in G.$$

Let $g \in N_G(QN) \cap (QN)^x = N_G(QN) \cap Q^x N$. Then $g = q^x n$ for some $q \in Q$ and $n \in N$, and as $n \in QN \leq N_G(QN)$, it follows that $q^x \in N_G(QN)$ and it suffices to show that $q^x \in QN$. Since $Q \in \mathrm{Syl}_p(QN)$, the number of conjugates of Q in QN is congruent to 1 modulo p and consequently q^x must fix a conjugate Q^m of Q in QN, where $m \in N$. Thus $q^{xm^{-1}} \in N_G(Q) \cap Q^{xm^{-1}}$ and as $Q \in \mathcal{H}(G)$, it follows that $q^{xm^{-1}} \in Q$, yielding $q^x \in Q^m \leq QN$, as required. $\square$

We need the following result of Shirong ([Shi, Theorems 1(2) and 2]):

Theorem 2.1.17 (Shirong [Shi])**.**

(1) *If P is a Sylow 2-subgroup of G and if every cyclic subgroup of P of order 2 or 4 is normal in $N_G(P)$, then G is 2-nilpotent.*

(2) *If G possesses a normal 2-complement N and if every minimal subgroup of any Sylow subgroup R of N is normal in $N_G(R)$, then G is supersolvable.*

Our first result follows easily from Theorem 2.1.17:

Theorem 2.1.18 [CsH, Theorem 8]**.** *If $\mathcal{S}p(G) \subseteq \mathcal{H}(G)$, then G is supersolvable.*

Proof. Let P be a Sylow p-subgroup of G and let $U \in \mathcal{S}p(P) \subseteq \mathcal{H}(G)$. Since $U \trianglelefteq\trianglelefteq N_G(P)$, it follows by Lemma 2.1.14(2) that $U \trianglelefteq N_G(P)$. Consequently, we may conclude that for every Sylow subgroup P of G, $\mathcal{S}p(P) \subseteq \mathcal{N}(N_G(P))$ holds. Thus, by Theorem 2.1.17(1), G possesses a normal 2-complement N and by Theorem 2.1.17(2) G is supersolvable, as claimed. $\square$

It is easy to see that even nilpotent groups do not necessarily satisfy the assumptions of Theorem 2.1.18. For example, if G is the dihedral group of order 8 and if $x \in G$ is a non-central involution, then $\langle x \rangle$ does not belong to $\mathcal{H}(G)$. So it appears of interest to characterize all groups satisfying the assumptions of Theorem 2.1.18.

For this characterization we need results about minimal non-supersolvable groups, i.e., non-supersolvable groups, all proper subgroups of which are supersolvable. Such groups were studied by Doerk [Do]. We shall use the following statement of these results by Shaalan [Sha, 2.15], which includes the extra property (5).

Theorem 2.1.19 [Sha]**.** *Let G be a minimal non-supersolvable group. Then:*

(1) *G has a non-trivial normal Sylow p-subgroup P for some prime p.*
(2) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*
(3) *If $p \neq 2$, then* $\exp(P) = p$.
(4) *If P is non-abelian and $p = 2$, then* $\exp(P) = 4$.
(5) *If P is abelian, then* $\exp(P) = p$.

We are now ready for the proof of

Theorem 2.1.20 [CsH, Theorem 10]**.** *$\mathcal{S}p(G) \subseteq \mathcal{H}(G)$ if and only if there exist subgroups L and K of G such that*

(1) *$G = L \rtimes K$.*
(2) *L and K are nilpotent Hall subgroups of G.*
(3) *$\mathcal{S}p(L) \subseteq \mathcal{N}(G)$.*
(4) *$\mathcal{S}p(K) \subseteq \mathcal{N}(K)$.*

Proof. Suppose first that $\mathcal{S}p(G) \subseteq \mathcal{H}(G)$. Then, by Theorem 2.1.18 G is supersolvable and hence G' is nilpotent. Let π denote the set of primes dividing $|G'|$, but not dividing $|G : G'|$, and let L denote the Hall π-subgroup of G'. We shall show in a number of steps that G/L is nilpotent. If $\pi = \pi(G')$, then $L = G'$ and our claim holds. So suppose that $\pi \subset \pi(G')$ and let $P \in \mathrm{Syl}_p(G)$, where $p \in \pi(G') - \pi$. Write $G' = (G' \cap P) \times T$.

Step 1. If q is a prime, $q \neq p$, and if $Q \in \mathrm{Syl}_q(N_G(P))$, then $Q \leq C_G(P)$.

Proof. If $p = 2$, then $q > p$ and the supersolvability of PQ implies that $Q \triangleleft PQ$. Since also $P \triangleleft PQ$, it follows that $Q \leq C_G(P)$, as claimed. So assume for the rest of this step that $p > 2$. Clearly $Q \leq N_G(G' \cap P)$ and each element of Q fixes the elements of $P/(G' \cap P)$ by conjugation. As $G' \cap P < P$, it follows by Huppert [Hu1, I.18.6] that there exists $x \in P - \{1\}$ such that $Q \leq C_G(x)$, and we may assume, without loss of generality, that $o(x) = p$. Since every minimal subgroup U of P belongs to $\mathcal{H}(G)$ and $U \trianglelefteq\trianglelefteq PQ$, it follows by Lemma 2.1.14(2) that $U \trianglelefteq PQ$ and $\Omega_1(P) \leq Z(P)$. Let $y \in \Omega_1(P) - \langle x \rangle$; then y and xy are also of order p and $Q \leq N_G(\langle y \rangle) \cap N_G(\langle xy \rangle)$. If z is an arbitrary element of Q, then $x^z = x$ and there exist integers i, j, $1 \leq i, j \leq p - 1$ such that $y^z = y^i$ and $(xy)^z = (xy)^j$. It follows that $xy^i = x^j y^j$ and consequently $y^{i-j} = x^{j-1}$. Since $\langle x \rangle \cap \langle y \rangle = \{1\}$, it follows that $i = j = 1$ and we may conclude that $Q \leq C_G(\Omega_1(P))$. Thus, by Huppert [Hu1, IV.5.12], $Q \leq C_G(P)$ as claimed.

Step 2. G/T is nilpotent.

Proof. Since $PT = G'P$, $PT/T \trianglelefteq G/T$ and $(G/T)/(PT/T)$ is an abelian p'-group. Let r be a prime dividing the order of $(G/T)/(PT/T)$. By the Frattini argument with respect to G and $G'P$, $G = G'N_G(P) = TN_G(P)$ and hence r divides $N_G(P)$.

Let $R \in \mathrm{Syl}_r(N_G(P))$; then $RT/T \in \mathrm{Syl}_r(G/T)$, and, by Step 1, RT/T centralizes PT/T. We may conclude that the abelian complement of PT/T in G/T centralizes PT/T, which implies that G/T is nilpotent, as claimed.

Step 3. G/L is nilpotent.

Proof. Repeating the process of Steps 1 and 2 with respect to all primes in $\pi(G')-\pi$, and noticing that L is the intersection of the corresponding T's we may conclude from Step 2 that G/L is nilpotent.

Now the first direction of the proof of Theorem 2.1.20 can be easily completed. Since L is a nilpotent Hall π-subgroup of G and G/L is nilpotent, it follows by the Schur–Zassenhaus theorem that there exists a nilpotent complement K of L in G, proving (1) and (2). Since $\mathcal{S}p(G) \subseteq \mathcal{H}(G)$, by Lemma 2.1.14(2) each $U \in \mathcal{S}p(G)$ is normal in every subgroup of G in which it is subnormal and (3), (4) follow easily.

Conversely, suppose that (1)–(4) are satisfied and let $U \in \mathcal{S}p(G)$. If $U \leq L$, then $U \trianglelefteq G$ by (3) and hence $U \in \mathcal{H}(G)$. If $U \leq K^t$ for some $t \in G$, then $U^{t^{-1}} \trianglelefteq K$ by (4) and hence $U \trianglelefteq K^t$, yielding $G = LN_G(U)$. As $(|U|, |L|) = 1$, we conclude by Lemma 2.1.15 that $U \in \mathcal{H}(G)$ and the proof is complete. □

Let $H \trianglelefteq G$ and suppose that G/H is supersolvable. Our next aim is to show that if H satisfies certain conditions, then also G is supersolvable.

Theorem 2.1.21 [CsH, Theorem 11]. *If $H \trianglelefteq G$, G/H is supersolvable and $\mathcal{S}p(H) \subseteq \mathcal{H}(G)$, then G is supersolvable.*

Proof. First we prove Theorem 2.1.21 under the assumption that H is a p-group for some prime p. Suppose that the result is false and let G be a counterexample of a minimal order. If $X < G$, then $(XH)/H \cong X/(X \cap H)$ is supersolvable and if $U \in \mathcal{S}p(X \cap H)$, then by our assumptions and Lemma 2.1.14(4), $U \in \mathcal{H}(G) \cap \mathcal{S}u(X) \subseteq \mathcal{H}(X)$. Thus, by the minimality of G, X is supersolvable, and it follows that G is a minimal non-supersolvable group. By Theorem 2.1.19 there exists a non-trivial Sylow q-subgroup P of G which is normal in G and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. Moreover, the exponent of P is q if $q > 2$ and it is either 2 or 4 if $q = 2$, which implies that $P = \langle \mathcal{S}p(P) \rangle$. If K denotes a complement of P in G, then $G/P \cong K$ is supersolvable and if $q \neq p$, then $P \cap H = 1$ and $G \cong G/(P \cap H) \lesssim G/P \times G/H$ is supersolvable, a contradiction. Hence $q = p$ and $H \leq P$. Since $H \trianglelefteq G$ and $P/\Phi(P)$ is minimal normal in $G/\Phi(P)$, it follows that either $H\Phi(P) = \Phi(P)$ or $H\Phi(P) = P$. In the former case $H \leq \Phi(P) \leq \Phi(G)$, whence, by our assumptions, $G/\Phi(G)$ and consequently also G are supersolvable, a contradiction. So we must have $H\Phi(P) = P$, which yields $H = P$. Thus, by our assumptions, $\mathcal{S}p(P) = \mathcal{S}p(H) \subseteq \mathcal{H}(G)$ and if $U \in \mathcal{S}p(P)$, then $U \lhd\lhd G$ and $U \lhd G$ by Lemma 2.1.14(2). Arguing as above, we conclude that either $U = P$ or

$U \leq \Phi(P)$. If $U = P$, then P is cyclic and G/P is supersolvable, which implies that G is supersolvable, a contradiction. Hence $U \leq \Phi(p)$ for each $U \in \mathcal{S}p(P)$ and $P = \langle \mathcal{S}p(p) \rangle \leq \Phi(P)$, a final contradiction, since $P > 1$.

Suppose now that H is arbitrary, and let again G be a counterexample of minimal order. By Lemma 2.1.14(4) $\mathcal{S}p(H) \subseteq \mathcal{H}(G) \cap \mathcal{S}u(H) \subseteq \mathcal{H}(H)$ and it follows by Theorem 2.1.18 that H is supersolvable. If p is the largest prime dividing $|H|$ and P is the Sylow p-subgroup of H, then $P \lhd G$. Consider G/P. If $U \in \mathcal{S}p(H/P)$, then $U = (QP)/P$, where $Q \in \mathcal{S}p(H) \subseteq \mathcal{H}(G)$ and $(|Q|, p) = 1$. Thus, by Lemma 2.1.16 $U \in \mathcal{H}(G/P)$. Moreover, $(G/P)/(H/P) \cong G/H$ is supersolvable and it follows by the minimality of G that G/P is supersolvable. Hence, as shown in the opening paragraph, G is supersolvable. We have reached our final contradiction and the proof of Theorem 2.1.21 is now complete. $\square$

If G is solvable then the condition for supersolvability used in Theorem 2.1.21 can be weakened as follows:

Theorem 2.1.22 [CsH, Theorem 12]**.** *If $H \trianglelefteq G$ is solvable, G/H is supersolvable and $\mathcal{S}p(F(H)) \subseteq \mathcal{H}(G)$, then G is supersolvable.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. Let p be the smallest prime in $\pi(F(H))$ and let $P \in \mathrm{Syl}_p(F(H))$. Clearly $P \lhd G$ and we denote $F(H/P) = L/P$. Let $P_1/P \in \mathrm{Syl}_p(L/P)$. As $P_1 \trianglelefteq H$, it follows that $P_1 \leq F(H)$ and $P_1 = P$. Hence $(p, |L/P|) = 1$ and $P \in \mathrm{Syl}_p(L)$. Since, by Lemma 2.1.14(4) and our assumptions, $\mathcal{S}p(P) \subseteq \mathcal{H}(G) \cap L \subseteq \mathcal{H}(L)$, it follows by Theorem 2.1.21 that L is supersolvable.

Suppose first that $p = 2$. Then $L = P \times T$, with $T \cong L/P$ a nilpotent group and it follows that $L \leq F(H)$. Thus $L/P \leq F(H)/P$, which implies $L/P = F(H)/P$. Let $V \in \mathcal{S}p(L/P)$. Then there exists $U \in \mathcal{S}p(F(H)) \subseteq \mathcal{H}(G)$ such that $V = (UP)/P$. Since $(|U|, |P|) = 1$, Lemma 2.1.16 implies that $V \in \mathcal{H}(G/P)$. Hence $\mathcal{S}p(L/P) \subseteq \mathcal{H}(G/P)$ and by the minimality of G, G/P is supersolvable. Hence G is supersolvable by Theorem 2.1.21, a contradiction. Similarly, the case $H = P$ leads to a contradiction. Thus we may assume that $p > 2$ and $H > P$.

Let $q \in \pi(L/P)$. Then $q \neq p$ and each Sylow q-subgroup of L/P can be represented as $(QP)/P$, where Q is a q-subgroup of L. By Lemma 2.1.14(3) and Theorem 2.1.21, QP is supersolvable.

If $q \in \pi(F(H))$, then $q > p$ and $QP = Q \times P$. But $QP \trianglelefteq H$, so $Q \lhd H$ and $Q \leq F(H)$. Hence $\mathcal{S}p(Q) \subseteq \mathcal{H}(G)$ and we may conclude, in view of Lemma 2.1.16, that every subgroup of L/P of order q belongs to $\mathcal{H}(G/P)$.

If $q \notin \pi(F(H))$, write $F(H) = P \times T$. Since $F(H)/P \leq L/P$ and $(|T|, |Q|) = 1$, it follows that $[Q, T] \leq P \cap T = 1$ and $Q \leq C_H(T)$. We shall use this fact later. If $U \in \mathcal{S}p(P) \subseteq \mathcal{H}(G)$, then $U \lhd \lhd QP$ and hence, by Lemma 2.1.14(2), $U \lhd QP$, yielding $Q \le N_H(U)$. Moreover, $U \lhd P$, which implies that $\Omega_1(P) \le Z(P)$. Let $a \in Q - \{1\}$ and suppose that there exists $c \in P - \{1\}$ of order p such that $c^a = c$. If $U = \langle u \rangle \in \mathcal{S}p(P) - \langle c \rangle$, then $a \in N_H(U)$ and since c, $u \in Z(P)$, $a \in N_H(\langle uc \rangle)$ as well. Thus there exist integers i, j, $1 \le i, j \le p-1$ such that $c^a = c$, $u^a = u^i$ and $(cu)^a = (cu)^j = c^j u^j$. It follows that $cu^i = c^j u^j$, whence $c^{j-1} = u^{i-j}$. As $U \cap \langle c \rangle = 1$, it follows that $i = j = 1$ and hence $a \in C_H(\Omega_1(P))$. By Huppert [Hu1, IV.5.12], $a \in C_H(P)$ and as $Q \le C_H(T)$, it follows by the solvability of H that $a \in C_H(F(H)) \le F(H)$, contradicting $q \notin \pi(F(H))$. So each nontrivial element of Q acts fixed-point-freely on P by conjugation. If U is a subgroup of P of order p, then $Q \le N_H(U)$ and hence Q is cyclic. Consequently, each $V \in \mathcal{S}p(L/P)$ of order q or 4 if $q = 2$ satisfies: $V \operatorname{char} (QP)/p \operatorname{char} H/P \lhd G/P$, so $V \lhd G/P$ and hence $V \in \mathcal{H}(G/P)$.

We may conclude that $\mathcal{S}p(F(H/P)) \subseteq \mathcal{H}(G/P)$ and it follows by the minimality of G that G/P is supersolvable. Thus, by Theorem 2.1.21, G is supersolvable and we have reached our final contradiction. The proof of Theorem 2.1.22 is complete.□

Theorems 2.1.21 and 2.1.22 immediately yield the following result concerning the supersolvator $\operatorname{sups}(G)$ of G.

Corollary 2.1.23 [CsH, Corollary 13]**.** *A group G is supersolvable under each of the following conditions:*

(1) $\mathcal{S}p(\operatorname{sups}(G)) \subseteq \mathcal{H}(G)$; *or*

(2) *G is solvable and* $\mathcal{S}p(F(\operatorname{sups}(G))) \subseteq \mathcal{H}(G)$.

Later Yangming Li [Ya] generalized the notion of $\mathcal{H}$-subgroups introducing the notion of NE-subgroups and using our results he obtained similar sufficient conditions for supersolvability.

M. Asaad [As2] introduced the concept of the weakly supersolvable embedded p-subgroups. Studying the $\mathcal{H}$-property of these before mentioned subgroups and using our results he also received necessary and sufficient conditions for supersolvability.

Requiring the $\mathcal{H}$-property of certain abelian subgroups M. Ramadan [Ra1] found sufficient conditions for supersolvability.

2.2. Supersolvability

Recall that a group is called supersolvable, if every its chief factor is cyclic.

In the first chapter we presented some sufficient conditions for supersolvability of finite groups supposing the S-quasinormality or $\mathcal{H}$-property of some minimal subgroups. In many cases these sufficient conditions seemed to be necessary too. In general we obtained the following: A group G is supersolvable if and only if there exists a normal subgroup H of G such that G/H is supersolvable and H must have certain required properties. In each case the proof of necessity was very simple: Supposing the supersolvability of G one can obviously put $H = 1$ and the trivial subgroup satisfies the required properties.

The other problem concerning these sufficient conditions is the following: In the previous chapter Theorems 2.1.9 and 2.1.10 give the structure of solvable groups provided that every subgroup of $F(G)$ of prime order or order 4 is S-quasinormal in G. And this structure really shows that these conditions describe only a relatively small subclass of supersolvable groups.

In this chapter our aim is to find proper sufficient and necessary conditions for supersolvability using a weakened version of S-quasinormality and $\mathcal{H}$-property.

My first results concerning this problem appeared in [Cs2]. Then I received sharper statements in [Cs1]. First we prove a natural factorization of supersolvable groups. The corollary of this theorem is another characterization of supersolvable groups based on the structure of the Fitting subgroup. By using these results we describe the structure of these subclasses of supersolvable groups obtained under the assumption that some subgroups satisfy certain conditions.

Main results. Huppert proved [Hu2, Satz 10.3, p. 724] the following theorem. If a finite group is the product of pairwise permutable cyclic subgroups, then it is supersolvable. Of course the converse of this statement is not even true in the class of nilpotent groups, since there are nonabelian groups of exponent p when $p > 2$. By studying it, we find that a supersolvable group can be decomposed as a product of cyclic subgroups of prime power order that are permutable if their orders are powers of different primes and those belonging to the same prime satisfy certain conditions.

Theorem 2.2.1 [Cs1, Theorem 1]**.** *Let G be a group with $\pi(G) = \{p_1, \dots, p_k\}$. Then G is supersolvable if and only if for all $p_i \in \pi(G)$ there is a Sylow p_i-subgroup P_i and cyclic subgroups P_{i_l} $(1 \le l \le t_i)$ of P_i such that*

(i) $P_i = P_{i_1} P_{i_2} \dots P_{i_{t_i}}$,

(ii) $P_{i_1} \dots P_{i_l} \lhd P_i$, *for all* $1 \le l \le t_i$,

(iii) $P_{i_l} \cdot P_{j_m} = P_{j_m} \cdot P_{i_l}$, *for all* $1 \le i,\ j \le k,\ i \ne j,\ 1 \le l \le t_i,\ 1 \le m \le t_j$.

Moreover, for every chief series refining a Sylow tower there exists such a factorization of Sylow subgroups as given above.

For the proof we need the following result.

Lemma 2.2.2 [Cs1, Lemma 1]**.** *Let A be an abelian normal Sylow p-subgroup of a group G. Let $a \in A$ be of order p such that $\langle a \rangle$ is normal in G and $A/\langle a \rangle$ is cyclic. Then either A is cyclic or $A = \langle a \rangle \times \langle b \rangle$, where $\langle b \rangle$ is normal in G.*

Proof. Assume that A is not cyclic. Obviously G/A acts on $A/\Phi(A)$. By Maschke's Theorem $A/\Phi(A) = \langle a\Phi(A) \rangle \times \langle b\Phi(A) \rangle$, where $\langle b\Phi(A) \rangle$ is G/A invariant. As A is abelian, $A = \langle a \rangle \langle b \rangle$ holds. Since $o(a) = p$ we conclude that here $|A : \langle b \rangle| = p$. Also $\langle b \rangle \supseteq \Phi(A)$, whence $\langle b \rangle$ is G/A-invariant. Consequently $\langle b \rangle \lhd G$. $\square$

Lemma 2.2.3 [Cs1, Lemma 2]**.** *Let G be a supersolvable group. Let P be a normal Sylow p-subgroup of G and H a p-complement. Then $P = A_1 \cdot A_2 \ldots A_s$, where every A_i is a cyclic subgroup of P such that $H \leq N_G(A_i)$ and $A_1A_2 \ldots A_i \lhd P$, for all $1 \leq i \leq s$.*

Proof. We prove our statement by induction on the order of P. The supersolvability of G implies that there exists a subgroup A_1 of P of order p such that A_1 is normal in G. We can assume that $A_1 \neq P$. Obviously G/A_1 satisfies the conditions of our lemma. By induction on G/A_1 there exist cyclic subgroups $D_2/A_1, \ldots, D_s/A_1$ of P/A_1 such that $HA_1/A_1 \leq N_{G/A_1}\ (D_i/A_1)$ and $(D_2/A_1) \ldots (D_s/A_1) \lhd P/A_1 = D_2(A_1) \ldots D_s(A_1)$, for all $2 \leq i \leq s$. Thus $H \leq N_G(D_i)$ and, since $A_1 \leq Z(P)$, the D_i are abelian. Applying Lemma 2.2.2 to D_i and A_1, we can see that for all i there is a cyclic subgroup A_i of D_i such that $D_i = A_1 \cdot A_i$ and A_i is normalized by H. The subgroups A_i have the required properties. $\square$

Proof of Theorem 2.2.1. First we assume the supersolvability of G. Then G possesses an ordered Sylow tower by [Hu2, Satz 9.1, p. 716]. Suppose $p_1 > p_2 > \cdots > p_k$. Then for each p_i we have a Sylow p_i-subgroup P_i such that $P_i \leq N_G(P_j)$ for all $j < i$. Using the normality of P_i in $P_iP_{i+1} \ldots P_k$ we apply Lemma 2.2.3 to P_i and $P_iP_{i+1} \ldots P_k$. We get a factorization $P_i = P_{i_1} \ldots P_{i_{t_i}}$, where P_{i_r} is a cyclic subgroup normalized by $P_{i+1} \ldots P_k$ and $P_{i_1} \ldots P_{i_r}$ is normal in P_i, for all $1 \leq r \leq t_i$. Let $1 \leq i,\ j \leq k,\ i \neq j,\ 1 \leq l \leq t_i,\ 1 \leq m \leq t_j$. Assume $i < j$. Then, as stated before, P_{i_l} is normalized by P_j, whence $P_{i_l}P_{j_m} = P_{j_m}P_{i_l}$ holds.

Conversely, assume that G is a group satisfying (i), (ii), and (iii) of the theorem. Suppose $p_1 > p_2 > \cdots > p_k$. Let $N = P_{1_1}$. By hypothesis we have $N \lhd P_1$ and $NP_{i_l} = P_{i_l}N$, for all $2 \leq i \leq k$ and $1 \leq l \leq t_i$. By Ito's Theorem [Hu2, Satz 10.1, p. 722] we get that NP_{i_l} is supersolvable whence, since $p_1 > p_i$, $N \lhd NP_{i_l}$ follows. Therefore $P_i \leq N_G(N)$, for all $2 \leq i \leq k$. Thus we conclude $N \lhd G$. Since G/N

obviously inherits the hypothesis, we have by induction that G/N is supersolvable. As N is cyclic, we find G is supersolvable. $\square$

M. Asaad and M. Ramadan in [AsR, Theorem 3.3] proved the following result. Suppose that G is solvable and the Frattini subgroup $\Phi(G) = 1$. Then G is supersolvable if and only if the Fitting subgroup $F(G)$ is the direct product of some normal subgroups of G of prime order.

Using our factorization on supersolvable groups we generalize the theorem above. Not supposing $\Phi(G) = 1$ we give another characterization of supersolvable groups. For this aim we introduce the following concept:

Definition. A subgroup H of G is called *weak S-quasinormal* in G if, for every $p \in \pi(G)$, there is at least one Sylow p-subgroup of G that permutes with H.

Later in the literature a similar notion appeared independently: the 3-permutability (or Σ-permutability): As we have seen in the previous chapter in case of 3-permutability we require the permutability of the given subgroup with all Sylow subgroups of the fixed Sylow system.

Remark. It follows from Theorem 2.2.1 that a supersolvable group is the product of some weak S-quasinormal cyclic subgroups of prime power orders.

Theorem 2.2.4 [Cs1, Theorem 2]**.** *For a group G the following statements are equivalent.*

(a) *G is supersolvable.*

(b) *$G' \leq F(G)$ and $F(G)$ is the product of cyclic and weak S-quasinormal subgroups of G of prime power orders.*

(c) *There is a nilpotent normal subgroup N of G, such that $G' \leq N$ and N is the product of cyclic and weak S-quasinormal subgroups of G of prime power orders.*

Proof. (a) $\Longrightarrow$ (b) Let G be supersolvable. Then by [Hu2, Satz 9.1, p. 716] G' is nilpotent, whence $F(G) \geq G'$. The supersolvability implies the existence of an ordered Sylow tower. Let $\pi(G) = \{p_1, \ldots, p_k\}$ with $p_1 > p_2 > \cdots > p_k$ and let $P_1P_2 \ldots P_i$ $(i = 1, \ldots, k)$ be a Sylow tower of G. Clearly there exists a chief series refining our Sylow tower such that it contains $P_1P_2 \ldots P_{i-1}(P_i \cap F(G))$ for all $1 \leq i \leq k$. Applying Theorem 2.2.1, it is easy to see that $P_i \cap F(G)$ is the product of weak S-quasinormal cyclic subgroups.

(b) $\Longrightarrow$ (c) This is trivial, because we may choose $N = F(G)$.

(c) $\Longrightarrow$ (a) Hypothesis (c) is obviously inherited by all quotient groups. Let G be a group of minimal order that is not supersolvable but satisfies (c). By the minimality we conclude that G has a unique minimal normal subgroup M. As G is solvable, $\Phi(G) = 1$ and $M = F(G)$. Obviously $N = M = G'$ is an elementary abelian p-group, for some prime p, and G/N is a p'-group, so that N is the Sylow p-subgroup of G. By the conditions $N = N_1 \cdot N_2 \cdot \ldots \cdot N_t$ with cyclic and weak S-quasinormal subgroup N_i of G. The weak S-quasinormality implies that for every $q \neq p$ there is a Sylow q-subgroup Q such that $QN_i = N_iQ$ for all $1 \leq i \leq t$. Since N_i is subnormal in G, obviously $N_i \lhd N_iQ$. As $N_i \lhd N$, we find $N_G(N_i) = G$, whence $N_i = N$. We have N_i is cyclic and consequently G is supersolvable, a contradiction. $\square$

We try to weaken these conditions to give another characterization of supersolvable groups.

Theorem 2.2.5 [Cs1, Theorem 3]**.** *Let G be a group with $G' \leq F(G)$. Then G is supersolvable if and only if there exists a normal subgroup H of G such that G/H is supersolvable and $F(H)$ is the product of cyclic and weak S-quasinormal subgroups of G.*

Proof. (1) Assume that G is supersolvable. We may choose $H = F(G)$. Using our Theorem 2.2.4, we conclude that H satisfies the conditions.

(2) Let G be a group of minimal order that is not supersolvable, but has got a normal subgroup H with the required properties. We now aim to show that $\Phi(G)=1$. Assume $\Phi(G) \neq 1$. Since $F(G) \geq G'$, G is solvable, whence $F(G) \neq \Phi(G)$. Clearly $F(G) \cap H = F(H)$. If $\Phi(G) \cap H = F(G) \cap H$, using again $F(G) \geq G'$, we conclude that $H\Phi(G)/\Phi(G)$ is abelian and $H\Phi(G)/\Phi(G) \cap F(G)/\Phi(G) = 1$. We have $H\Phi(G)/\Phi(G) \lhd G/\Phi(G)$ and further $F(G/\Phi(G)) = F(G)/\Phi(G)$. As G is solvable, $C_{G/\Phi(G)}(F(G/\Phi(G))) \leq F(G/\Phi(G))$. It follows that $H \leq \Phi(G)$, whence $G/\Phi(G)$ is supersolvable. Using Huppert's Theorem [Hu2, Satz VI.8.6] we get that G is supersolvable, contradicting the minimality of G. Thus $\Phi(G) \cap H \neq F(G) \cap H = F(H)$. Obviously $G/\Phi(G)$ satisfies the conditions of our theorem. The minimality of G yields the supersolvability of $G/\Phi(G)$. Using again Huppert's Theorem we find that G is supersolvable, a contradiction. Thus $\Phi(G) = 1$.

The supersolvability of G/H implies the existence of the following chain: $F(G) \cap H = F(H) = F_0 \lhd F_1 \lhd \ldots \lhd F_k = F(G)$ such that $F_i \lhd G$ and F_i/F_{i-1} is of prime order for all $1 \leq i \leq k$. Assume F_i/F_{i-1} is of order p. Let H be a Hall subgroup of G with $\pi(H) = \pi(G) \setminus \{p\}$. Then H acts on F_i and F_{i-1}. Using Maschke's Theorem we get $F_i = F_{i-1} \times \langle b_i \rangle$ and $H \leq N_G(\langle b_i \rangle)$ so that $\langle b_i \rangle$ is weak S-quasinormal in G. As we have $F(G) \cap H$ is the product of cyclic weak S-quasinormal subgroups of G, we conclude that $F(G)$ is the product of cyclic weak S-quasinormal subgroups of G. Applying our Theorem 2.2.4 we find that G is supersolvable. This is the final contradiction. $\square$

Now we are ready to prove Theorem 2.1.10 from the previous chapter. First we repeat this theorem.

Theorem 2.1.10 [Cs1, Theorem 4]**.** *Let G be a supersolvable group and P is a normal p-subgroup of G with $p \neq 2$. Then every minimal subgroup of P is normal in P and S-quasinormal in G if and only if there is a chain $1 = P_0 \triangleleft P_1 \triangleleft \ldots \triangleleft P_k = P$ with $P_i \triangleleft G$, $|P_i/P_{i-1}| = p$ for every $1 \leq i \leq k$ and $\Omega_1(P) = P_\ell \leq Z(P)$, for some $1 \leq \ell \leq k$. Moreover, for every $g \in G$ with $(o(g), p) = 1$, there exists a natural number t_g with $1 \leq t_g \leq p-1$ such that $a^g = a^{t_g}$, where a is an arbitrary element of $D = \bigtimes_{i=1}^{k}(P_i/P_{i-1})$.*

Proof of Theorem 2.1.10. Assume that every minimal subgroup of P is normal in P and S-quasinormal in G. Then these minimal subgroups are clearly in $Z(P)$; that is $\Omega_1(P) \leq Z(P)$. By the supersolvability of G there is a chain $1 = P_0 \triangleleft P_1 \triangleleft \ldots \triangleleft P_k = P$ such that $P_i \triangleleft G$ and P_i/P_{i-1} is of order p, for all $1 \leq i \leq k$. Moreover $P_l = \Omega_1(P)$, for some $1 \leq l \leq k$. Let $g \in G$ with $g \neq 1$, $(o(g), p) = 1$ and $u \in \Omega_1(P)$, $u \neq 1$. As $\langle u \rangle$ is subnormal in G, and g is the product of elements of prime power orders, the S-quasinormality implies $g \in N_G(\langle u \rangle)$. Thus we have $u^g = u^{t_g}$, for some natural number t_g with $1 \leq t_g \leq p-1$. Let $v \in \Omega_1(P)$, $1 \neq v \neq u$. Similarly $v^g = v^l$ where $1 \leq l \leq p-1$. Since $(uv)^g = (uv)^k$, for some natural number $1 \leq k \leq p-1$, using $\Omega_1(P) \leq Z(P)$ we find $k = t_g = l$. Thus $w^g = w^{t_g}$, for all $w \in \Omega_1(P)$. Apply our factorization Theorem 2.2.1 to $P\langle g \rangle$ and to the chief series $1 = P_0 \triangleleft P_1 \triangleleft \ldots \triangleleft P_k = P \triangleleft P\langle g \rangle$. Then for every $1 \leq i \leq k$ there exists $a_i \in P_i$ such that $P_{i-1}\langle a_i \rangle = P_i$ and $g \in N_G(\langle a_i \rangle)$. Suppose $o(a_i) = p^{m_i}$ and ${a_i}^g = {a_i}^{t_i}$, for some natural number $1 \leq t_i \leq p^{m_i}-1$. Since $(a_i^{p^{m_i-1}})^g = (a^{p^{m_i-1}})^{t_g}$ we conclude $t_i \equiv t_g(p)$. Moreover $\bar{a}_i^g = \bar{a}_i^{t_g}$ is true for all $1 \leq i \leq k$ and $\bar{a}_i \in P_i/P_{i-1}$. Consequently $a^g = a^{t_g}$ follows for an arbitrary element a of $D = \bigtimes_{i=1}^{k}(P_i/P_{i-1})$.

Assume conversely that g has got a chain with the required properties. As $\Omega_1(P) \leq Z(P)$, every minimal subgroup of P is normal in P. Let $g \in G$ with $g \neq 1$, $(o(g), p) = 1$. Applying Lemma 2.2.3 to $\Omega_1(P)\langle g \rangle$, we get $\Omega_1(P) = \langle b_1 \rangle \times \cdots \times \langle b_l \rangle$ with $g \in N_G(\langle b_i \rangle)$ for all i. Using the hypothesis ${b_i}^g = {b_i}^{t_g}$ follows, whence $b^g = b^{t_g}$ holds for every $b \in \Omega_1(P)$. It easily follows from this fact that every minimal subgroup of P is S-quasinormal in G. $\square$

Later in [ACs3] with M. Asaad we improved and extended this natural factorization and other theorems concerning sufficient and necessary conditions for supersolvability obtained in [Cs1]. We rely on Σ-permutability and the generalized Fitting subgroup.

We assume throughout this section that $\mathcal{F}$ is a saturated formation. Each group G has a smallest normal subgroup $G^{\mathcal{F}}$ such that $G/G^{\mathcal{F}}$ is in $\mathcal{F}$. This uniquely determined normal subgroup $G^{\mathcal{F}}$ of G is called the $\mathcal{F}$-residual subgroup of G. Let $\mathcal{U}$ denote the class of all supersolvable groups. It is well known that $\mathcal{U}$ is a saturated formation.

We say Σ is a Sylow system of a group G, if Σ contains exactly one Sylow subgroup for every prime dividing the order of G. A subgroup H of G is called Σ-quasinormal if H permutes with every element of the Sylow system Σ of G.

The object of this section is to study factorized groups whose subgroup factors are cyclic of prime power order and are connected by certain permutability properties. More precisely, we prove the following results:

Theorem 2.2.6 [Acs3, Theorem 1.1]**.** *The following two statements are equivalent:*

(1) $G \in \mathcal{U}$.

(2) *There is a normal subgroup H of G such that $G/H \in \mathcal{U}$ and there exists a Sylow system Σ of G such that for every $P_i \in \Sigma$, $P_i \cap H$ is the product of cyclic Σ-quasinormal subgroups.*

Theorem 2.2.7 [ACs3, Theorem 1.2]**.** *Let G be a group. Then the following two statements are equivalent:*

(1) $G \in \mathcal{U}$.

(2) *There exists a Sylow system Σ of G such that for every $P_i \in \Sigma$, $P_i^* = F^*(G) \cap P_i$ is the product of Σ-quasinormal cyclic subgroups and $G' \leq F^*(G)$.*

Theorem 2.2.8 [ACs3, Theorem 1.3]**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Then the following statements are equivalent:*

(1) $G \in \mathcal{F}$.

(2) *There exist a Sylow system of G, say $\Sigma = \{G_{p_1}, \ldots, G_{p_n}\}$ and a normal subgroup H of G such that $G/H \in \mathcal{F}$ and $F^*(H) \cap G_{p_i}$, where $i = 1, \ldots, n$, satisfies the following:*

(a) *$F^*(H) \cap G_{p_i} = A_{i_1} \ldots A_{i_{t_i}}$, where A_{i_ℓ} is a cyclic and Σ-quasinormal subgroup of G for every $1 \leq \ell \leq t_i$.*

(b) *$A_{i_1} A_{i_2} \ldots A_{i_\ell} \trianglelefteq G_{p_i}$ for all $1 \leq \ell \leq t_i$.*

For the proof of these theorems we need the following lemmas:

Lemma 2.2.9 [ACs3, Lemma 2.1]**.** *Assume $G \in \mathcal{U}$. Let P be a p-subgroup of G and K be a subgroup of $N_G(P)$ such that $(|K|, p) = 1$. Then $P = C_1 \cdot C_2 \cdots C_\ell$ where every C_i is a cyclic subgroup of P such that $K \leq N_G(C_i)$ for all $1 \leq i \leq \ell$.*

Proof. We prove our statement by induction on the order of G. The supersolvability of G implies there exists a normal subgroup $N = \langle a \rangle$ of order p_1 of G, where p_1 is the maximal prime divisor of the order of G. We can assume $N \neq P$. Obviously G/N satisfies every condition of our lemma. By induction on the order of G, there exist cyclic subgroups B_1/N, B_2/N, $\ldots$, B_ℓ/N of $PN/N \leq G/N$ such that KN/N normalizes B_i/N for every $1 \leq i \leq \ell$ and $PN/N = \prod_{i=1}^{\ell} B_i/N$.

a) Suppose $p_1 = p$. Then clearly $K \leq N_G(B_i)$, B_i is a p-subgroup for every $1 \leq i \leq \ell$ and $P = B_1 \cdot B_2 \cdots B_\ell$. Since $N \leq Z(B_i)$ it follows every B_i is abelian.

Suppose B_i is not cyclic for some $1 \leq i \leq \ell$. Obviously K acts on $B_i/\Phi(B_i)$ by conjugation. Using Maschke's theorem we get

$$B_i/\Phi(B_i) = \langle a\Phi(B_i)\rangle \times \langle b_i\Phi(B_i)\rangle$$

where $\langle a\rangle = N$ and $\langle b_i\Phi(B_i)\rangle$ is K-invariant. Since $|N| = p$ we can conclude $|B_i : \langle b_i\rangle| = p$. Also $\langle b_i\rangle \geq \Phi(B_i)$. Hence $\langle b_i\rangle$ is K-invariant. Thus $B_i = N \times D_i$ where $D_i = \langle b_i\rangle$ is a cyclic subgroup normalized by K.

b) Suppose $p_1 \neq p$. Then clearly $PN = B_1 \cdot B_2 \cdots B_\ell$ and $K \leq N_G(B_i)$ for every $1 \leq i \leq \ell$. It is easy to see that $B_i = (P \cap B_i)N$ where $P \cap B_i$ is cyclic for every $1 \leq i \leq \ell$ and $P = \prod_{i=1}^{\ell}(P \cap B_i)$. As $K \leq N_G(B_i) \cap N_G(P)$ we can conclude $K \leq N_G(P \cap B_i)$. □

Lemma 2.2.10 [ACs3, Lemma 2.2]**.** *Let G be a group. Let be $\pi(G) = \{p_1, \ldots, p_k\}$, where $p_1 > p_2 > \cdots > p_k$. Then the following two statements are equivalent:*

a) *$G \in \mathcal{U}$.*

b) *There exists a Sylow system Σ of G such that for every Sylow p_i-subgroup P_i of Σ there are cyclic subgroups P_{i_ℓ} $(\ell = 1, \ldots, t_i)$ of P_i such that*

i) *$P_i = P_{i_1} \ldots P_{i_{t_i}}$,*

ii) *$P_{i_n} \cdot P_{j_m} = P_{j_m} \cdot P_{i_n}$ for all $1 \leq i, j \leq k$, $i \neq j$, $1 \leq n \leq t_i$, $1 \leq m \leq t_j$.*

Proof. a) $\Longrightarrow$ b) The supersolvability of G implies the existence of an ordered Sylow tower by [Hu2, Satz 9.1, p. 716]. Let $\pi(G) = \{p_1, \ldots, p_k\}$ with $p_1 > p_2 > \cdots > p_k$. Then for each p_s there exists a Sylow p_s-subgroup P_s such that $P_s \leq N_G(P_\ell)$ for every $\ell < s$. Since $L_{s+1} = P_{s+1}P_{s+2}\ldots P_k \leq N_G(P_s)$, follows from Lemma 2.2.9 that $P_s = P_{s_1}\ldots P_{s_{t_s}}$ where P_{s_r} is a cyclic subgroup such that $L_{s+1} \leq N_G(P_{s_r})$ for every $1 \leq r \leq t_s$. Let $1 \leq i, j \leq k$, $i \neq j$, $1 \leq n \leq t_i$, $1 \leq m \leq t_j$. Assume $i < j$. Then clearly $P_j \leq N_G(P_{i_n})$ whence $P_{i_n}P_{j_m} = P_{j_m} \cdot P_{i_n}$.

b) $\Longrightarrow$ a) Our hypothesis is obviously inherited by all quotient groups. Let G be a group of minimal order that satisfies our hypothesis, but G is not supersolvable. Then, by Huppert, it follows $\Phi(G) = 1$. Using [Hu2, IV. Satz 2.8], we have $P_2\ldots P_k \leq N_G(P_{1_r})$ for every $1 \leq r \leq t_1$. Hence $\Phi(P_1) \leq \Phi(G) = 1$ and we can conclude P_{1_1} is a normal subgroup of G of order p_1. By the minimality of G, we have G/P_{1_1} is supersolvable, which implies the supersolvability of G, a contradiction. □

Lemma 2.2.11. *Let $H \trianglelefteq G$, $\pi(G) = \{p_1, \ldots, p_k\}$, $p_1 > p_2 > \cdots > p_k$. Let Σ be a Sylow system of G. Let $P_i \in \Sigma$, $i = 1, \ldots, k$. If $P_i \cap H$ is the product of Σ-quasinormal cyclic subgroups, then $H \in \mathcal{U}$.*

Proof. Put $P_i^* = P_i \cap H = P_{i_1} \dots P_{i_{t_i}}$, where $P_{i_{\ell_i}}$ is cyclic and Σ-quasinormal subgroup of G for every $1 \le \ell_i \le t_i$. We show that if $P_j^* = P_j \cap H = P_{j_1} \dots P_{j_{t_j}}$, where $i \ne j$, then $P_{i_{\ell_i}} \cdot P_{j_{\ell_j}} = P_{j_{\ell_j}} \cdot P_{i_{\ell_i}}$ for every $1 \le \ell_i \le t_i$, $1 \le \ell_j \le t_j$. By hypothesis $P_{i_{\ell_i}} P_j = P_j P_{i_{\ell_i}}$ and $P_{j_{\ell_j}} P_i = P_i P_{j_{\ell_j}}$. Let $P_{i_{\ell_i}} = \langle a \rangle$, $P_{j_{\ell_j}} = \langle b \rangle$. So $\langle a \rangle P_j = P_j \langle a \rangle$ and $\langle b \rangle P_i = P_i \langle b \rangle$. Now $ab = b_1 a^t$ with $b_1 \in P_j$, $a^t \in \langle a \rangle$, $ab = b^\ell a_1$ with $b^\ell \in \langle b \rangle$, $a_1 \in P_1$. Hence $b_1^{-1} b^\ell = a^t a_1^{-1} \in P_i \cap P_j = 1$ $(i \ne j)$. This implies $a^t = a_1$, $b^\ell = b_1$, i.e. $a_1 \in \langle a \rangle$, $b_1 \in \langle b \rangle$. Therefore our H satisfies the conditions of Lemma 2.2.10, hence H is supersolvable. $\square$

Proof of Theorem 2.2.6. (1) $\implies$ (2) It is clear by Lemma 2.2.10.

(2) $\implies$ (1) Let G be a counterexample of minimal order. Our hypothesis is obviously inherited by all quotient groups. Then all quotient groups are supersolvable by our minimal choice of G. Hence if $\Phi(G) \ne 1$, $G/\Phi(G)$ is supersolvable and so G is supersolvable by [We, Chapter 1, Corollary 3.2], a contradiction. Thus $\Phi(G) = 1$. We know by [DH, Theorem 6.10(c), p. 36] that $F(G)$ is the direct product of abelian minimal normal subgroups of G. By Lemma 2.2.11, H is supersolvable. Then R is characteristic in H, where $R \in \mathrm{Syl}_r(H)$ and r is the largest prime dividing $|H|$, so $R \trianglelefteq G$. Clearly $R \le F(H) \le F(G)$. If L and M are two distinct minimal normal subgroups of G, then G/L and G/M are supersolvable by our minimal choice of G and hence $G \cong G/L \cap M$ is supersolvable, a contradiction. Thus $F(G)$ is a unique minimal normal subgroup of G and $R = F(H) = F(G)$.

Let p_n be the largest prime dividing $|G|$ and let $P_n \in \mathrm{Syl}_{p_n}(G)$ such that $P_n \in \Sigma$. Clearly HP_n is a subgroup of G. Assume first that $HP_n < G$. Then HP_n is supersolvable by our minimal choice of G. Hence P_n is characteristic in HP_n. Also $HP_n \trianglelefteq G$ because G/H is supersolvable. So $P_n \trianglelefteq G$ and consequently $P_n = R = F(H) = F(G)$.

Our hypothesis implies that $P_n = R$ is the product of subgroups of order r and those subgroups are Σ-quasinormal in G. Then $|F(H)| = |F(G)| = r$ because $F(H)$ is the unique minimal normal subgroup of G. So G is supersolvable. Now assume that $HP_n = G$. We have the following two cases:

Case 1. $p_n/|H|$. Then $p_n = r$ and $P_n \cap H$ is a Sylow r-subgroup of H. Since H is supersolvable, it follows that $P_n \cap H$ is characteristic in H. Also $H \trianglelefteq G$ and so $P_n \cap H \trianglelefteq G$. Hence $P_n \trianglelefteq G$, because $G/P_n \cap H$ is supersolvable by our minimal choice of G. Thus $G = HP_n$ is supersolvable.

Case 2. $p_n \nmid |H|$. Then $P_{n-1} = F(H) = F(G)$, where $p_{n-1} = r$ is the largest prime dividing $|H|$. By hypothesis, $P_{n-1} = F(H) = F(G) = L_1 \times \cdots \times L_m$, where $|L_i| = p_{n-1} = r$, and L_i is Σ-quasinormal in G. Now it follows easily that $L_i \trianglelefteq G$ for all $1 \le i \le m$ and hence $L_i = P_{n-1}$ and so G is supersolvable. $\square$

Corollary 2.2.12. *The group $G \in \mathcal{U}$ if and only if there exists a Sylow system Σ*

of G such that for every $P_i \in \Sigma$, $P_i \cap G^{\mathcal{U}}$ is the product of Σ-quasinormal cyclic subgroups.

Proof of Theorem 2.2.7. (1) $\Longrightarrow$ (2) The supersolvability of G implies $F^*(G) = F(G)$ by [HuB, X.13], further G' is nilpotent, whence $G' \leq F(G)$ follows. From the supersolvability of G we know the existence of an ordered Sylow tower. Let $\pi(G) = \{p_1, \ldots, p_k\}$ with $p_1 > p_2 > \cdots > p_k$ and let $P_1P_2 \ldots P_i$ ($i = 1, \ldots, k$) be a Sylow tower, where $P_i \in \mathrm{Syl}_{p_i}(G)$. Let $\sum = \{P_1, \ldots, P_k\}$. We have $P_i^* = P_i \cap F^*(G) = P_i \cap F(G)$. Since $F(G)$ is nilpotent it follows ${P_i}^*$ is normal in G for every $1 \leq i \leq k$. Applying Lemma 2.2.9 for ${P_i}^*$ and $K = P_1P_2 \ldots P_{i-1}P_{i+1} \ldots P_k$ we get our statement.

(2) $\Longrightarrow$ (1) By Hypothesis $G' \leq F^*(G)$, so $G/F^*(G)$ is abelian. Hence G is supersolvable by Theorem 2.2.6. □

The formation $\mathcal{U}$ of all supersolvable groups is locally defined by the integrated full system $\{\mathcal{U}(p)\}$, where for each prime p, $\mathcal{U}(p)$ is the class of strictly p-closed groups. The $\mathcal{U}$-hypercenter of a group G is the largest supersolvably embedded subgroup $Q_\infty(G)$ of G (see [We]).

Lemma 2.2.13. *Let P be a normal p_1-subgroup of G, and let $\Sigma = \{G_{p_1}, \ldots, G_{p_n}\}$ be a Sylow system of G. If $P = A_1A_2 \ldots A_s$ where A_i is a cyclic Σ-quasinormal subgroup in G and $A_1A_2 \ldots A_i \trianglelefteq G_{p_1}$ for every $1 \leq i \leq s$, then $P \leq Q_\infty(G)$.*

Proof. Clearly $(A_1A_2 \ldots A_i)G_{p_j}$ is a subgroup of G for all $i = 1, \ldots, s$ and $j = 2, \ldots, n$. Also $A_1A_2 \cdot \ldots \cdot A_i \trianglelefteq (A_1 \ldots A_i)G_{p_j}$ ($i \neq j$) and since $A_1 \ldots A_i \trianglelefteq G_{p_i}$, we have $A_1 \ldots A_i \trianglelefteq G$ for every $i = 1, \ldots, s$. Now we have the following series:

$$1 \leq A_1 \leq A_1A_2 \leq \cdots \leq A_1A_2 \ldots A_{s-1} \leq A_1A_2 \ldots A_s = P,$$

where $A_1A_2 \ldots A_i/A_1 \ldots A_{i-1}$ is cyclic and $A_1 \ldots A_i \trianglelefteq G$ for all i. Therefore every chief factor of G which lies in P has prime order, so $P \leq Q_\infty(G)$. □

Lemma 2.2.14. *Let Σ be a Sylow system of G, where $\Sigma = \{G_{p_1}, \ldots, G_{p_k}\}$. Assume for every $G_{p_i} \in \Sigma$, $F^*(G) \cap G_{p_i} = A_{i_1}A_{i_2} \ldots A_{i_{t_i}}$ where A_{i_ℓ} is a cyclic and Σ-quasinormal subgroup of G for every $1 \leq \ell \leq t_i$. Assume further $A_{i_1}, \ldots, A_{i_s} \trianglelefteq G_{p_i}$ for every $1 \leq s \leq t_i$. Then $G \in \mathcal{U}$.*

Proof. By Lemma 2.2.11 $F^*(G)$ is supersolvable. Hence, by [HuB, X.13], it follows $F^*(G) = F(G)$. By Lemma 2.2.13, $F(G) \leq Q_\infty(G)$. Then $G/C_G(F(G))$ is supersolvable by [We, Theorem 7.15]. By [HuB, X.13], $C_G(F^*(G) \leq F(G)$ and since $F^*(G) = F(G)$, we have $C_G(F(G)) \leq F(G)$. Since $G/C_G(F(G))$ is supersolvable, we have $G/F(G)$ is supersolvable, and since $F(G) \leq Q_\infty(G)$, it follows easily that G is supersolvable. □

Proof of Theorem 2.2.8. (1) $\Longrightarrow$ (2) We may choose $H = 1$.

(2) $\Longrightarrow$ (1) By Lemma 2.2.14 H is supersolvable. By [HuB, X.13], $F^*(H) = F(H)$. Lemma 2.2.13 implies $F(H) \leq Q_\infty(G)$. Then $G/C_G(F(H)) \in \mathcal{U}$ by [We, Theorem 7.15] and since $G/H \in \mathcal{F}$ it follows $G/C_H(F(H)) \in \mathcal{F}$. But $C_H(F^*(H)) = C_H(F(H)) \leq F(H)$ by [HuB, X.13], therefore $G/F(H) \in \mathcal{F}$. Since $\mathcal{F}$ and $\mathcal{U}$ are saturated formations such that $\mathcal{U} \subseteq \mathcal{F}$ it follows from [DH, IV. Prop. 3.11 p. 362] that $Q_\infty(G) \leq Z_\mathcal{F}(G)$, so $F(H) \leq Z_\mathcal{F}(G)$ and since $G/F(H) \in \mathcal{F}$, we have that $G \in \mathcal{F}$. $\square$

Finally in this section we deal with groups satisfying the assumption that all their Sylow subgroups are abelian (SSA-groups, in short). Using a restatement of a characterization of supersolvable groups and the notion of $\mathcal{H}$-subgroups of G, we characterize supersolvable SSA-groups by the structure of their Sylow subgroups and by properties of the Fitting subgroup.

We start with a restatement of Theorem 1.2.1:

Theorem 2.2.15 [CsH, Theorem 17]**.** *Let G be a group with $\pi(G) = \{p_1, p_2, \dots, p_k\}$, where $p_1 > p_2 > \cdots > p_k$. Then G is supersolvable if and only if for each i, $1 \leq i \leq k$, there exists a Sylow p_i-subgroup P_i of G and cyclic subgroups P_{i_l}, $1 \leq l \leq t_i$ of P_i, such that*

(1) $P_i = P_{i_1} P_{i_2} \dots P_{i_{t_i}}$.
(2) $P_{i_1} P_{i_2} \dots P_{i_l} \trianglelefteq P_i$ *for* $1 \leq l \leq t_i$, *and*
(3) $P_j \leq N_G(P_{i_l})$ *for all* $j > i$ *and* $1 \leq l \leq t_i$.

Our aim is now to characterize supersolvable SSA-groups by the structure of their Sylow subgroups.

Theorem 2.2.16 [CsH, Theorem 18]**.** *Suppose that G is an SSA-group. Then G is supersolvable if and only if every Sylow subgroup of G is a product of cyclic $\mathcal{H}$-subgroups of G.*

Proof. Let $\pi(G) = \{p_1, p_2, \dots, p_k\}$, where $p_1 > p_2 > \cdots > p_k$ and let $|G| = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$.

Suppose first that G is supersolvable. Then G satisfies the properties listed in Theorem 2.2.15. In particular, by (3), $G = P_1 P_2 \dots P_k$, where $P_i \in \mathrm{Syl}_{p_i}(G)$ for $i = 1, \dots, k$ and $P_j \leq N_G(P_i)$ for all $j > i$. Consider P_{i_j} with $1 \leq i \leq k$ and $1 \leq l \leq t_i$. Then $P_i P_{i+1} \dots P_k \leq N_G(P_{i_l})$ and $M = P_1 P_2 \dots P_{i-1} \triangleleft G$. Thus $G = M N_G(P_{i_l})$ and by Lemma 2.1.15, P_{i_l} is an $\mathcal{H}$-subgroup of G. Hence each P_i is a product of cyclic $\mathcal{H}$-subgroups of G, as claimed.

Conversely, suppose that every Sylow subgroup of G is a product of cyclic $\mathcal{H}$-subgroups of G. We wish to show first that for $1 \leq i \leq k$ there exist Sylow p_i-subgroups P_i of G, such that $P_j \leq N_G(P_i)$ for all $j > i$. Denote $p_k = q$ and let $Q \in$

$\mathrm{Syl}_q(G)$. By our assumptions $Q = Q_1Q_2 \dots Q_s$, where Q_i are cyclic $\mathcal{H}$-subgroups of G for all i. Since $Q_i \lhd\lhd N_G(Q)$, it follows by Lemma 2.1.14(2) that $Q_i \trianglelefteq N_G(Q)$ for all i. There exists $K < N_G(Q)$ such that $(|K|, |Q|) = 1$ and $N_G(Q) = QK$. Since q is the smallest prime of $\pi(G)$ and Q_i are cyclic, $K < C_G(\Omega_1(Q_i))$ and by [Go, Theorem 5.2.4], $K < C_G(Q_i)$ for all i. Thus $K < C_G(Q)$, which implies that $N_G(Q) = C_G(Q)$ and by Burnside's theorem G possesses a normal q-complement M_{k-1}. Every Sylow subgroup of M_{k-1} is also a product of cyclic $\mathcal{H}$-subgroups of G, which, by Lemma 2.1.14(3), are $\mathcal{H}$-subgroups of M_{k-1} as well. If $r = p_{k-1}$, then the above argument implies that M_{k-1} possesses a normal r-complement M_{k-2} and continuing this process we obtain a series of normal Hall-subgroups of G

$$M_0 = 1 < M_1 < M_2 < \cdots < M_{k-2} < M_{k-1} < M_k = G$$

with $|M_i : M_{i-1}| = p_i^{r_i}$ for $1 \leq i \leq k$. Let $P_1 = M_1$. Then $P_1 \trianglelefteq G$ and denote a complement of P_1 in G by C_2. By Lemma 2.1.14(3) every Sylow subgroup of C_2 is a product of cyclic $\mathcal{H}$-subgroups of C_2 and it follows by the above arguments that the Sylow p_2-subgroup P_2 of C_2 is normal in C_2. Let C_3 denote a complement of P_2 in C_2. Then the Sylow p_3-subgroup P_3 of C_3 is normal in C_3. By continuing this process we obtain Sylow p_i-subgroups P_i of G for $1 \leq i \leq k$, such that $P_j \leq N_G(P_i)$ for all $j > i$, as claimed above.

By our assumption, for each i, $1 \leq i \leq k$, there exist cyclic $\mathcal{H}$-subgroups P_{i_l}, $1 \leq l \leq t_i$, such that $P_i = P_{i_1}P_{i_2} \dots P_{i_{t_i}}$. Thus condition (1) of Theorem 2.2.15 is satisfied, and in order to prove that G is supersolvable, it remains to show that conditions (2) and (3) of Theorem 2.2.15 also hold with respect to these subgroups of G. Since the P_i are abelian, (2) certainly holds. If $j > i$, then $P_j \leq N_G(P_i)$ and it follows that $P_{i_l} \lhd\lhd P_iP_j$. Hence by Lemma 2.1.14(2), $P_{i_j} \trianglelefteq P_iP_j$, yielding (3). Thus by Theorem 2.2.15, G is supersolvable and the proof is complete. $\square$

Using Theorems 2.2.15 and 2.2.16 we wish now to give another characterization of supersolvable SSA-groups, refering to a decomposition of $F(G)$ into a product of cyclic $\mathcal{H}$-subgroups of G.

Theorem 2.2.17 [CsH, Theorem 19]**.** *Let G be an SSA-group. Then the following statements are equivalent:*

(1) *G is supersolvable.*

(2) *$G' \leq F(G)$ and $F(G)$ is a product of cyclic $\mathcal{H}$-subgroups of G of prime power orders, and*

(3) *There exists a nilpotent normal subgroup N of G such that $G' \leq N$ and N is a product of cyclic $\mathcal{H}$-subgroups of G of prime power orders.*

Proof. (1) $\implies$ (2) Let G be supersolvable with $\pi(G) = \{p_1, p_2, \dots, p_k\}$, where $p_1 > p_2 > \cdots > p_k$. Then $G' \leq F(G)$ and by Theorem 2.2.15 there exist Sylow

p_i-subgroups P_i of G, $1 \le i \le k$, such that $G_i = P_1P_2 \dots P_{i-1}(P_i \cap F(G))P_{i+1} \dots P_k$ is a subgroup of G for all i. By Theorem 2.2.16, $P_i \cap F(G)$ is a product of cyclic $\mathcal{H}$-subgroups of G_i, which are also $\mathcal{H}$-subgroups of G, since P_i is abelian. This proves (2).

(2) $\Longrightarrow$ (3) We may choose $N = F(G)$.

(3) $\Longrightarrow$ (1) By induction on $|G|$. Let $N = N_1N_2 \dots N_k$, where N_i are non-trivial cyclic subgroups of N which belong to $\mathcal{H}(G)$. Since $N \trianglelefteq G$ and N is nilpotent, it follows by Lemma 2.2.14(2) that $N_i \trianglelefteq G$ for $i = 1, \dots, k$. Let $R = N_1$ and consider G/R. Then $N/R = ((N_2R)/R) \dots (N_kR)/R))$, with $N_iR \trianglelefteq G$ for $i = 2, \dots, k$. Hence $N_iR \in \mathcal{H}(G)$, which implies by Lemma 2.2.14(1) that $(N_iR)/R \in \mathcal{H}(G/R)$ for $i = 2, \dots, k$. It follows that G/R inherits the assumptions of (3) with respect to N/R and by induction G/R is supersolvable. Since R is cyclic, G is supersolvable as well, as required. $\square$

Finally we show that an SSA-group G, with $F(G)$ and G' cyclic, is supersolvable.

Corollary 2.2.18 [CsH, Corollary 20]**.** *Let G be an SSA-group and suppose that $F(G)$ is cyclic and contains G'. Then G is supersolvable.*

Proof. Since $F(G) \trianglelefteq G$, it is a cyclic $\mathcal{H}$-subgroup of G and hence G is supersolvable by Theorem 2.2.17. $\square$

Later M. Asaad in [As1] improved and extended our characterization concerning SSA-groups, i.e. our Theorems 2.2.16 and 2.2.17. He got a similar sufficient condition for the supersolvability but requiring the commutativity of only the Sylow 2-subgroups.

2.3. T-groups, T^*-groups

T-groups

A group G is called a T-group if every its subnormal subgroup is normal in G. The study of T-groups is relatively old, the first result having been found in 1896 by R. Dedekind [De]. What Dedekind did was to determine all the finite groups in which every subgroup is normal, a result later extended by R. Baer [Bae1] to infinite group. The motivation for Dedekind's work was algebraic number theory: he wished to determine the algebraic number fields with the property that every subfield is normal.

Today groups with all their subgroups normal are called Dedekind groups: they are either abelian or direct product of a quaternion group of order 8 and an abelian group without elements of order 4.

The first explicit mentioning of T-groups in the literature is in the paper of E. Best and O. Taussky [BT] from 1942. They showed that any group with cyclic Sylow subgroups is a T-group. Subsequently G. Zacher [Za] characterized solvable T-groups by means of Sylow tower properties:

Theorem 2.3.1 (Zacher [Za]). *Let G be a solvable group, $p_1 > p_2 > \cdots > p_k$ are the prime divisors of the order of G and $S_1, \ldots, S_k$ is a Sylow system with $S_i \in \mathrm{Syl}_{p_i}(G)$. Then G is a T-group if and only if the following statements are true:*

(i) *S_i is an abelian or a Hamiltonian group $(1 \leq i \leq k)$.*

(ii) *If $1 \leq i < j \leq k$, then $S_j \leq N_G(S_i)$.*

(iii) *If $1 \leq i < j \leq k$ and $y \in S_j$, then there exists $n \in \mathbb{Z}$ such that for every $x \in S_i$ $yxy^{-1} = x^n$.*

The decisive structure theorem for solvable T-group was proved by W. Gaschütz [Ga] in 1957:

Theorem 2.3.2. *A group G is a solvable T-group if and only if it has an abelian normal Hall subgroup L of odd order such that G/L is a Dedekind group and elements of G induce power automorphisms in L.*

We propose to call the nilpotent residual of G *the nilpotator of G* and denote it by $\mathrm{nilp}(G)$. Thus $\mathrm{nilp}(G)$ denotes the intersection of all normal subgroups N of G with G/N nilpotent. It is well known that $\mathrm{nilp}(G) = L$, where L is the smallest term of the lower central series of G.

The nilpotator of G plays an important role in the following result of Gaschütz:

Theorem 2.3.3 (Gaschütz [Ga]). *Let G be a solvable T-group and let $L = \mathrm{nilp}(G)$. Then:*

(1) *L is an abelian Hall subgroup of G of odd order; and*

(2) *G/L is a Dedekind group.*

Since Dedekind groups of odd order are abelian and $\mathrm{nilp}(G) \leq G'$, Theorem 2.3.3 and Feit–Thompson's theorem [FT] immediately imply

Corollary 2.3.4 [CsH, Corollary 2]. *Let G be a T-group of odd order. Then $G' = \mathrm{nilp}(G)$ is an abelian Hall subgroup of G.*

The next result is a combination of a result from Gaschütz [Ga] and the main results of Bianchi et al. [BMHV, Theorems 1 and 10]:

(I remind the reader that $\mathcal{H}(G)$ is the set of $\mathcal{H}$-subgroups of G. A subgroup K of G is called an $\mathcal{H}$-subgroup of G, if $N_G(K) \cap K^g \subseteq K$ for all $g \in G$.)

Theorem 2.3.5. *The following statements are equivalent:*

(1) *G is a solvable T-group;*

(2) *G is a supersolvable T-group;*

(3) *All subgroups of G belong to $\mathcal{H}(G)$; and*

(4) *All subgroups of G of prime power order belong to $\mathcal{H}(G)$.*

The aim of this section is to give a structural characterization of these groups, based on the assumption that certain subgroups of G belong to $\mathcal{H}(G)$. In particular, if G is of odd order, then G is a T-group if and only if G' is a Hall subgroup of G and every subgroup of G' of prime power order belongs to $\mathcal{H}(G)$.

We start with

Theorem 2.3.6 [CsH, Theorem 14]. *Let G be a solvable group. Then G is a T-group if and only if there exists a subgroup L of G such that*

(1) *L is a normal Hall subgroup of G;*

(2) *G/L is a Dedekind group, and*

(3) *every subgroup of L of prime power order belongs to $\mathcal{H}(G)$.*

Proof. If G is a solvable T-group, then, by Theorem 2.3.3, $L = \mathrm{nilp}(G)$ is a normal Hall-subgroup of G with a Dedekind quotient group and by Theorem 2.3.5 each subgroup of G belongs to $\mathcal{H}(G)$, implying (3). Conversely, suppose that G is solvable and conditions (1)–(3) are satisfied. By Theorem 2.3.5, it suffices to prove that every p-subgroup of G belongs to $\mathcal{H}(G)$, and, by our assumptions, it suffices to prove it for all primes dividing $|G : L|$. Since L is a normal Hall-subgroup of G with a Dedekind quotient group, there exist a Dedekind complement K of L such that $G = LK$ and $L \cap K = 1$. If P is a p-subgroup of G with p dividing $|G : L|$, then we may assume that $P \leq K$, which implies that $K \leq N_G(P)$ and $G = LN_G(P)$. Hence, by Lemma 2.1.15, $P \in \mathcal{H}(G)$ and the proof is complete. $\square$

If G is a solvable group of odd order, then being a T-group is equivalent to certain properties of G' and its subgroups.

Corollary 2.3.7 [CsH, Corollary 15]**.** *Let G be a solvable group of odd order. Then G is a T-group if and only G' is a Hall subgroup of G and every subgroup of G' of prime power order belongs to $\mathcal{H}(G)$.*

Proof. If G is a solvable T-group of odd order, then, by Corollary 2.3.4, $G' = \mathrm{nilp}(G)$ is a Hall-subgroup of G and by Theorem 2.3.5 all subgroups of G belong to $\mathcal{H}(G)$, implying the claimed statement. Conversely, if the conditions on G' are satisfied, then G is a T-group by Theorem 2.3.6. $\square$

By the Feit–Thompson theorem, the assumption concerning the solvability of G may be dropped in Corollary 2.3.7. Our last result is similar to Theorem 2.3.6, but here L denotes a nilpotent Hall subgroup of G.

Corollary 2.3.8 [CsH, Corollary 16]**.** *G is a solvable T-group if and only if there exist subgroups L and K of G such that*

(1) *$G = L \rtimes K$;*
(2) *L is a nilpotent Hall subgroup of G;*
(3) *K is a Dedekind group, and*
(4) *every subgroup of L of prime-power order belongs to $\mathcal{H}(G)$.*

Proof. If G is a solvable T-group, then , by Theorem 2.3.3, (1)–(3) are satisfied with respect to $L = \mathrm{nilp}(G)$ and a complement K of L in G. Moreover, by Theorem 2.3.5, each subgroup of G belongs to $\mathcal{H}(G)$ and hence also (4) is satisfied. Conversely, if conditions (1)–(4) hold, then G is a solvable group and by Theorem 2.3.6 it is a T-group, as claimed. $\square$

T^*-groups

Next we recall that a subgroup H is said to be permutable (or quasinormal) in a group G if $HK = KH$ for all subgroups K of G, i.e. if it permutes with every subgroup of G. Permutability can be considered thus as a weak form of normality.

A group is called PT-group, if the permutability is transitive, that is, H is permutable in K and K is permutable in G, then H is permutable in G. According to a theorem of Ore [Or] every permutable subgroup of a group is subnormal. Consequently PT-groups are exactly those groups in which subnormality and permutability coincide that is those groups in which every subnormal subgroup permutes with every other subgroup.

The problem is the following, what would happen if we did not require that every subnormal subgroup of a group permutes with any other subgroup of G but only with a certain family of its subgroups. I remind the reader that a subgroup of a group G is said to be S-quasinormal (or π-quasinormal), if it permutes with every Sylow subgroup of G.

A group is called T^*-group or PST-group if S-quasinormality is transitive, i.e. H is S-quasinormal in K, and K is S-quasinormal in G, then H is S-quasinormal in G. According to a theorem of Kegel [Ke] every S-quasinormal subgroup is subnormal.

Thus a group G is a T^*-group (or PST-group) if every its subnormal subgroup is S-quasinormal (π-quasinormal) in G.

The structure of solvable PST-group first was determined by R. Agrawal [Ag]:

Theorem 2.3.9. *A group G is a solvable PST-group (T^*-group) if and only if it has an abelian normal subgroup L of odd order such that G/L is nilpotent and elements of G induce power automorphisms in L.*

Gaschütz [Ga] showed that a subgroup of a solvable T-group G is a T-group. Peng [Pe] showed that G is a solvable T-group if and only if every p-subgroup is pronormal in G for all primes p. We say that a subgroup H of a group G is pronormal in G if and only if for all x in G, H and H^x are conjugate in $\langle H, H^x\rangle$. Along the same lines, the paper [Cs9, Theorem 1] established the following result: Let G be a group of odd order. Then G is a solvable T-group if and only if each subgroup of prime power order of G' is pronormal in G, and G' is a π-Hall subgroup of G.

By replacing the notion of T-groups with T^*-groups, we prove theorems which extend the above menitoned results.

Originally in our paper [ACs4] we used the word π-quasinormality, but here in every statement S-quasinormality will be used.

Lemma 2.3.10 [ACs4, Lemma 1]**.** *Let G be a solvable T^*-group. Then*

(i) *factor groups of G are solvable T^*-groups,*

(ii) *G is supersolvable.*

Proof. If $H \leqq K \leqq G$ and $K \triangleleft G$, then H is S-quasinormal in G if and only if K / H is S-quasinormal in G / K. Hence, factor groups of G are solvable T^*-groups and (i) holds.

(ii) Let L be a minimal normal subgroup of G. Since G is solvable, it follows that L is an elementary abelian p-group for some prime p dividing $|G|$. Let P be a Sylow p-subgroup of G such that $L \leqq P$. Then $L \cap Z(P) \neq 1$. Let L_1 be a subgroup of $L \cap Z(P)$ of order p. Clearly L_1 is subnormal in G. Hence L_1 is S-quasinormal in G by hypothesis. Let q be any prime in $\pi(G) \setminus \{p\}$ and let Q be any Sylow q-subgroup of G. Then L_1Q is a subgroup of G as L_1 is S-quasinormal in G. Since L_1 is a subnormal Hall subgroup of L_1Q, we have that $L_1 \lhd L_1Q$ and so $Q \leqq N_G(L_1) \leqq G$. But then

$$O^p(G) = \langle Q/Q \text{ is a Sylow } q\text{-subgroup of } G,\, q \neq p\rangle \leqq N_G(L_1)$$

and since $P \leqq N_G(L_1)$, it follows that $L_1 \lhd G$. Then by (i) $G\,/\,L_1$ is a solvable T^*-group and hence $G\,/\,L_1$ is supersolvable by induction on $|G|$. Therefore, G is supersolvable. □

Remarks. 1) Nilpotent groups are solvable T^*-groups. The converse is not true. This is easily seen by examining S_3, the symmetric group of degree 3.

2) Solvable T^*-groups are supersolvable (see Lemma 2.3.10). The converse is not true. Consider the group $G = S_3 \times Z_3$. Clearly, G is supersolvable; but there exists a subnormal subgroup H of G of order 3 such that H is not S-quasinormal in G. Thus G is not a T^*-group.

3) Solvable T-groups are solvable T^*-groups. The converse is not true. Let G be a nonabelian nilpotent group of odd order. Then by 1), G is a solvable T^*-group. If G is a solvable T-group, then G is abelian by Theorem 12.5.4 of [Ha], a contradiction. Thus G is not a solvable T-group.

4) Here is a summary of relationships between several important classes of solvable groups.

$$\begin{gathered}\text{Abelian groups} \subset \text{solvable } T\text{-groups} \subset \text{solvable } T^*\text{-groups}\\ \subset \text{supersolvable groups} \subset \text{solvable groups.}\end{gathered}$$

Also, we have

$$\begin{gathered}\text{Nilpotent groups} \subset \text{solvable } T^*\text{-groups}\\ \subset \text{supersolvable groups} \subset \text{solvable groups.}\end{gathered}$$

Each class is contained in, but different from, the class following it.

We need the following lemmas.

Lemma 2.3.11 [ACs4, Lemma 2]**.** *Let Q be a q-subgroup of a finite group G and let R be a Sylow r-subgroup of G ($r \neq q$) such that $R \leqq N_G(Q)$. Assume $RQ_1 = Q_1R$ for some subgroup Q_1 of Q. Then $R \leqq N_G(Q_1)$.*

Proof. It is trivial. □

Lemma 2.3.12 [ACs4, Lemma 3]**.** *Let G be a finite group. Let U be a subgroup of prime power order of G, and let V be a subgroup of G such that $(|V|, |U|) = 1$ and V normalizes every subgroup of U. Assume there exists a proper normal subgroup U_1 of U such that each element of V induces the identity on U/U_1 by conjugation. Then $V \leqq C_G(U)$.*

Proof. Applying a theorem of Glauberman [Gla] $C_U(V)U_1 = U$ follows. Denote $L = C_U(V)$. Assume $L \lneqq U$. Clearly there exists an element a of U such that $a \notin LUU_1$. Consider $M = \langle a \rangle$. As $V \leqq N_G(M)$ and each element of V induces the identity on U/U_1 by conjugation, consequently each element of V induces the identity on $M/U_1 \cap M$. Applying again the theorem of Glauberman [Gla] $C_M(V)(U_1 \cap M) = M$ follows. Clearly $C_M(V) \leqq L$. Let r be the minimal positive integer such that $a^r \in C_M(V)$ and $a^r a^{-1} \in U_1$. Let t be the minimal positive integer such that $a^t \in U_1$. Obviously $r = m \cdot t + 1$. As U is a p-subgroup for some prime p, t and r are divisible by p, a contradiction. Thus $L = U$ i.e. $V \leqq C_G(U)$ is true. $\square$

Lemma 2.3.13 [ACs4, Lemma 4]**.** *Let P be a Sylow p-subgroup of a group G. Assume every subgroup of P is S-quasinormal in $N_G(P)$. Then either $P \leqq G'$ or each Sylow q-subgroup ($q \neq p$) of $N_G(P)$ centralizes P.*

Proof. Assume $P \nleqq G'$. Denote $P \cap G' = S$. Clearly $S \lhd N_G(P)$. Let $q \in \pi(N_G(P)) \setminus p$. Let Q be an arbitrary Sylow q-subgroup of $N_G(P)$. Obviously each element of Q induces an automorphism of P by conjugation, and clearly they induce the identity on P/S. Q normalizes every subgroup of P by Lemma 2.3.11. Then $Q \leqq C_G(P)$ by Lemma 2.3.12. $\square$

Theorem 2.3.14 [ACs4, Theorem 1]**.** *If G is a solvable T^*-group then all its subgroups are solvable T^*-groups too.*

Proof. Suppose that the theorem is false and let G be a counter-example of smallest order. Then there exists a proper subgroup H of G such that H is not a T^*-group. Let P be a Sylow p-subgroup of G where p is the largest prime dividing the order of G. Since G is supersolvable G has a Sylow tower and P is normal in G.

($*$) We show that for every subgroup P^* of P and Sylow r-subgroup R ($r \neq p$) of G, R normalizes P^*.

Clearly $R \leqq N_G(P)$. As G is a T^*-group and P^* is subnormal in G, $RP^* = P^*R$. Then $R \leqq N_G(P^*)$ by Lemma 2.3.11.

Our choice of G implies that every subgroup of G/P is a T^*-group. Hence if $P \cap H = 1$, $PH/P \cong H$ would be a T^*-group. Thus $P_0 = P \cap H \neq 1$. Clearly P_0 is normal in H and it is a Sylow p-subgroup L of H. Since H is not a T^*-group it follows that there exists a subnormal subgroup of H such that L is not S-quasinormal in H.

Assume $P_0 \leqq L$. PH/P is a T^*-group but $PH/P = H/P_0$, hence L/P_0 is S-quasinormal in H/P_0. Let A be an arbitrary Sylow subgroup of H. Then $L/P_0 \cdot AP_0/P_0 = AP_0/P_0 \cdot L/P_0$. Consequently $LAP_0 = AP_0L$, whence $LA = AL$ follows, which contradicts the fact that L is not S-quasinormal in H. Thus $P_0 \nleqq L$. Denote $P_1 = L \cap P_0$.

Assume $L = P_1$. By $(*)$ every Sylow r-subgroup ($r \neq p$) of H normalizes P_1, whence P_1 is S-quasinormal in H, a contradiction. So $L \neq P_1$. The supersolvability of G (see Lemma 2.3.10) implies that L is supersolvable, hence there exists a p-complement K in L, i.e. $L = K \cdot P_1$ and $K \cap P_1 = 1$. As L is subnormal in H, there is a chain $L \lhd L_1 \lhd L_2 \lhd \ldots \lhd L_k = H$. Let L_i be such that $P_0 \leqq L_i$, but $P_0 \nleqq L_{i-1}$. Since $(|K|, |P_0|) = 1$, K normalizes every subgroup of P_0 by $(*)$. Since $L_{i-1} \lhd L_i$, $P_0 \lhd L_i$ and $K \leqq L_{i-1}$, it follows that each element of K induces the identity on $P_0/P_0 \cap L_{i-1}$ by conjugation. Using Lemma 2.3.12, $K \leqq C_G(P_0)$ is true.

Let B be an arbitrary Sylow subgroup of H. Since H/P_0 is a T^*-group $LBP_0 = BL$ follows, consequently $LP_0B = BLP_0$. As P_0 is normal in H, LP_0 is a subgroup of H, thus LP_0B is a subgroup in H. Since $L = KP_1$, $LP_0 = KP_0$ and $LP_0B = KP_0B$ are true. Denote $M = KP_0B$. Denote N a p-complement of M such that $N \geqq K$. So $M = P_0N$. By Sylow's theorem there exists an element x of P_0 such that $B^x \leqq N$. We have $M = KP_0B$ and $|M| = \frac{|KP_0|\,|B|}{|KP_0 \cap B|}$. As $KP_0 = K \times P_0$, $KP_0 \cap B = K \cap B$ follows, furthermore $|M| = \frac{|K|\,|P_0|\,|B|}{|K \cap B|}$ is true. Since $M = P_0N$, $|M| = |P_0|\,|N|$. By the above $N = \frac{|K|\,|B|}{|K \cap B|}$. As $x \in P_0$ and $P_0 \leqq C_G(K)$ consequently $K \cap B = K \cap B^x$ and $|N| = \frac{|K|\,|B^x|}{|K \cap B^x|}$. We have $B^x \leqq N$, so $N = K \cdot B$. Thus $K \cdot B^x = B^x \cdot K$. Since $x \in P_0$ and $P_0 \leqq C_G(K)$, $KB = BK$ is true. Let B^* be a Sylow subgroup of G such that $B^* \geqq B$. As P_1 is subnormal in G consequently $P_1B^* = B^*P_1$. Let $a \in P_1$, $b \in B$ be arbitrary, then $ab = b_1a_1$ with $b_1 \in B^*$, $a_1 \in P_1$ but $ab \in L$, $a_1 \in L$ whence $b_1 \in L \cap B^* = B$. So $P_1B \leqq BP_1$, similarly $BP_1 \leqq P_1B$ is true. Hence $P_1B = BP_1$. Since $L = KP_1$, $LB = BL$ follows. Thus L is S-quasinormal in H, a contradiction.

$\square$

Theorem 2.3.15 [ACs4, Theorem 2]. *G is a solvable T^*-group if and only if $G = HK$ where H is a nilpotent normal Hall subgroup of G, K is a nilpotent Hall subgroup of G, $H \cap K = 1$, furthermore for arbitrary $x \in H$, $y \in K$ there exists an integer i such that $x^y = x^i$.*

Proof. 1) First we prove that the structure of a solvable T^*-group is the one described above.

Let $\pi(G) = \{p_1, \ldots, p_n\}$ with $p_1 > p_2 > \cdots > p_n$.

As G is supersolvable for all $1 \leqq i \leqq n$, there is a Sylow p_i-subgroup P_i such that if $1 \leqq k < j \leqq n$ then $P_j < N_G(P_k)$.

As G is a solvable T^*-group $N_G(P_i)$ is a T^*-group too by Theorem 2.3.14. Consequently every subgroup of P_i is S-quasinormal in $N_G(P_i)$. Lemma 2.3.13 implies that $P_i \leqq G'$ or each Sylow p_r-subgroup ($r \neq i$) of $N_G(P_i)$ centralizes P_i.

By the supersolvability of G, G' is nilpotent. Let H be the direct product of every P_j, $1 \leqq j \leqq n$ such that $P_j \leqq G'$. So H is a nilpotent normal Hall subgroup of G. Let P_k, P_m, $1 \leqq k$, $m \leqq n$, $k \neq m$ be such that $P_k, P_m \nleqq G'$. Assume $k < m$, then P_m is a Sylow subgroup of $N_G(P_k)$. By the above P_m centralizes P_k.

Let K be the direct product of every P_h with $1 \leqq h \leqq n$ and $P_h \not\leqq G'$. Clearly K is a nilpotent Hall subgroup of G furthermore $G = HK$, $H \cap K = 1$. As every subgroup of prime power order of G' is subnormal in G, applying Lemma 2.3.11 for arbitrary $x \in H$, $y \in K$ we can find an integer i such that $x^y = x^i$.

2. Assume that the structure of G is the one described in the statement of our theorem. Thus $G = HK$, $K \cap H = 1$. H is a nilpotent normal Hall subgroup of G, K is a nilpotent Hall subgroup of G furthermore for arbitrary $x \in H$, $y \in K$ there exists an integer i such that $x^y = x^i$.

Let B be a subnormal subgroup of G. We show B is S-quasinormal in G.

Using the solvability of G, Hall's theorem and the structure of G we can assume $B = H_1K_1$ with $K_1 \leqq K$ and $H_1 \leqq H$.

Let S be an arbitrary Sylow subgroup of G.

a) $S \leqq H$. Clearly $S \lhd G$, consequently $BS = SB$ is true.

b) $S \not\leqq H$. By Sylow's theorem $S^{z_0} \leqq K$ for some $z_0 \in G$. Since $G = HK$, $z_0 = z \cdot z_1$ with $z_1 \in K$, $z \in H$. By the nilpotency of K, $S^{z_0} = S^z$ and $K_1 \leqq N_G(S^z)$.

Let Q be a Sylow subgroup of H with $Q \not\leqq B$. As B is subnormal in G, there is a chain $B \lhd B_1 \lhd \ldots \lhd G$. Obviously there exists an index i such that

$$B_i > 0, \quad \text{such that} \quad B_{i-1} \not> Q.$$

Since K_1 normalizes Q, $K_1 < B_{i-1}$ and $B_{i-1} \lhd B_i$ it follows that each element of K_1 induces the identity on $Q/Q \cap B_{i-1}$. As $(|Q|, |K_1|) = 1$ and K_1 normalizes every subgroup of Q, hence $K_1 \leqq C_G(Q)$ by Lemma 2.3.12.

Let V be the direct product of the Sylow subgroups Q of H such that $Q \not\leqq B$. Then $H = W \times V$, consequently $z = w \times v$ with $w \in W$, $v \in V$. Hence $S^z = S^{wv}$. By the above $K_1 \leqq C_G(v)$, whence $K_1 \leqq N_G(S^w)$ follows. As $w \in B$, $K_1^{\omega^{-1}} \leqq B$ is true. Using the structure of G, it is clear that S normalizes H_1. As $B = K_1^{\omega^{-1}} \cdot H_1$, $BS = SB$ is true.

Thus G is a solvable T^*-group. □

Theorem 2.3.16 [ACs4, Theorem 3]**.** *Let G be a solvable group. If G is not a T^*-group and all proper subgroups of G are T^*-groups, then $|\pi(G)| = 2$.*

Proof. Let G be a counter-example of smallest order. Then $|\pi(G)| \geqq 3$. Since G is not a T^*-group, it follows that G possesses a subnormal subgroup L of G such that L is not S-quasinormal in G. Among all such L, choose L of minimal order. By hypothesis, L is a solvable T^*-group and so L is supersolvable by Lemma 2.3.10. Hence L possesses a normal subgroup L_1 of order p, where p is the largest prime dividing $|L|$. Hence if L_1 is a proper subgroup of L, our choice of L implies that L_1 is S-quasinormal in G. If L_1 is normal in G, then all proper subgroups of $G\,/\,L_1$

are T^*-groups and hence G / L_1 is a T^*-group or not. If G / L_1 is a T^*-group, then L / L_1 is subnormal in G / L_1 and hence L / L_1 is S-quasinormal in G / L_1 and so L is S-quasinormal in G, a contradiction. If G / L_1 is not a T^*-group, then our choice of G implies that $|\pi(G / L_1)| = 2$ and since $|\pi(G)| \geqq 3$, it follows that L_1 is a Sylow p-group of G. Hence $G / L_1 \cong K$, where K is a p'-Hall subgroup of G. But $K \cong G / L_1$ is a T^*-group, a contradiction. Thus L_1 is not normal in G. Consider

$$O^p(G) = \langle Q \text{ is a Sylow } q\text{-subgroup of } G,\ q \neq p \rangle.$$

Then $O^p(G) \leqq N_G(L_1) < G$. Obviously, all proper subgroups of G are supersolvable and $|\pi(G)| \geqq 3$. Hence if G is supersolvable, $R \triangleleft G$, where R is a Sylow r-subgroup of G and r is the largest prime dividing $|G|$. Also, if G is a minimal non-supersolvable group, then $R \triangleleft G$ by Doerk [Do]. Hence if $r = p$, $LO^p(G) \leqq N_G(L_1) < G$. By hypothesis, $LO^p(G)$ is a T^*-group and since L is subnormal in $LO^p(G)$, it follows that for any prime q in $\pi(G) \setminus \{p\}$ and any Sylow q-subgroup of G, LQ is a subgroup of G, and since LR is a subgroup of G, it follows that L is S-quasinormal in G, a contradiction. Thus $r \neq p$ and so $r \nmid |L|$. Consider LR. It is a subgroup of G and since L is subnormal Hall subgroup of LR, it follows that $LR = L \times R$. Let K be a p'-Hall subgroup of G such that $L \leqq K$. Since $G = RK$, it follows that $L^g = L^{hk}$, $g = hk$, $h \in R$, $k \in K$, $L^g = L^k \leqq K$ and so $L^G \leqq K$. Let Q be any Sylow q-subgroup of G, $q \neq r$. Then $L^G Q$ is a subgroup of G. Obviously, $L^G Q$ is a proper subgroup of G and so $L^G Q$ is a T^*-group by hypothesis. Since L is subnormal in $L^G Q$ it follows that L is S-quasinormal in $L^G Q$ and hence LQ is a subgroup of G and since $R \triangleleft G$, we have that L is S-quasinormal in G, a contradiction. Now we may assume that L is of prime order. If $|L| = r$, where r is the largest prime dividing $|G|$, then $\langle L, Q \rangle \leqq RQ$, where Q is any Sylow q-subgroup of G, $q \neq r$. Since $|\pi(G)| \geqq 3$, it follows that RQ is a proper subgroup of G and so RQ is a T^*-group. Since L is subnormal in RQ, L is S-quasinormal in RQ and so $\langle L, Q \rangle = LQ = QL$. Also $LR = RL = R$. Therefore, L is S-quasinormal in G, a contradiction. Thus $|L| = p$, where $p < r$ and r is the largest prime dividing $|G|$. Let P be a Sylow p-subgroup of G such that $L \leqq P$. Since L is not S-quasinormal in G, it follows easily that $N_G(L) < G$. For any prime q in $\pi(G) \setminus \{p\}$, there always exists a Sylow q-subgroup Q of G such that PQ is a subgroup of G as G is solvable. Since PQ is a proper subgroup of G, PQ is a solvable T^*-group and so L is S-quasinormal in PQ. Thus LQ is a subgroup of G. By Kegel [Ke] L is S-quasinormal Hall in LQ, so $L \triangleleft LQ$. Hence for any q in $\pi(G) \setminus \{p\}$, $Q \leqq N_G(L) < G$. We may assume without loss of generality that $N \leqq N_G(L) < G$, where N is a p'-Hall subgroup of G. Clearly, $G = PN_G(L)$. Now $L^g = L^{ab}$, $a \in N_G(L)$, $b \in P$, $g = ab$, implies that $L^b \leqq P$ and so $L^G \leqq P$. Hence for any prime q in $\pi(G) \setminus \{p\}$, it follows that $L^G Q$ is a proper subgroup of G, where Q is any Sylow q-subgroup of G. By hypothesis, $L^G Q$ is a solvable T^*-group and since L is subnormal in $L^G Q$, L is S-quasinormal in $L^G Q$

and so LQ is a subgroup of G. On the other hand, $L \leqq L^G \leqq P$ and so L lies in any Sylow p-subgroup P_1 of G, i.e., $P_1L = LP_1 = P_1$. Therefore, L is S-quasinormal in G, a final contradiction. □

By Feit–Thompson [FT], it is well known that a group of odd order is solvable. Now we can prove

Corollary 2.3.17 [ACs4, Corollary 4]**.** *If G is not a T^*-group and all proper subgroups of G are T^*-groups, then $|\pi(G)| = 2$.*

Proof. Let p be the smallest prime dividing $|G|$ and let P be a Sylow p-subgroup of G. Suppose that G has no normal p-complement. By Theorem 14.4.7 of [Ha], there is a subgroup P_1 of P which is normalized but not centralized by an element x of order prime to p, then $P_1\langle x\rangle$ is solvable. Hence if $P_1\langle x\rangle = G$, we are ready by Theorem 2.3.16. Thus $P_1\langle x\rangle$ is a proper subgroup of G and so $P_1\langle x\rangle$ is a solvable T^*-group. But then $P_1\langle x\rangle$ is supersolvable and so $P_1\langle x\rangle = P_1 \times \langle x\rangle$, and this is impossible. Thus G has a normal p-complement. Now the Feit–Thompson Theorem [FT] implies that G is solvable and so $|\pi(G)| = 2$ by Theorem 2.3.16. □

Corollary 2.3.18 [ACs4, Corollary 5]**.** *If G is not a T^*-group and all proper subgroups of G are T^*-groups, then there exist a Sylow p-subgroup P of G and a Sylow q-subgroup Q of G for some distinct primes p and q such that* (i) *$G = PQ$,* (ii) *$P \lhd G$, and Q is cyclic.*

Proof. By Corollary 2.3.17 (i) holds. By Burnside's Theorem, G is solvable and so all proper subgroups of G are solvable T^*-groups. Hence all proper subgroups of G are supersolvable and so there exists a normal Sylow p-subgroup P of G, say, such that $P \lhd G$ by Doerk [Do]. Thus (ii) holds. If $\langle y\rangle$ is a proper subgroup of Q, then $P\langle y\rangle$ is a solvable T^*-group. By Theorem 2.3.15, for arbitrary $x \in P$, there exists an integer i such that $x^y = x^i$. Hence if Q is not cyclic, G is a solvable T^*-group by Theorem 2.3.15 and this is impossible. Thus Q is cyclic and (ii) holds. □

Theorem 2.3.19 [ACs4, Theorem 6]**.** *If G/H is a solvable T^*-group, where H is a normal Hall subgroup of G, and all subgroups of prime power order of H are pronormal in G, then G is a solvable T^*-group.*

Proof. We use induction on $|G|$. By Peng's Theorem [Pe], H is a solvable T-group, so H is a solvable T^*-group. Hence if $H = G$, G is a solvable T^*-group and we are ready. Thus $H < G$. Let M be a maximal subgroup of G. If $H \nleqq M$, then $G = HM$, so $G/H \cong M/M \cap H$. Hence M is a solvable T^*-group by induction on $|G|$. If $H \leqq M$, then $M/H \leqq G/H$, so M/H is a solvable T^*-group by Theorem 2.3.14. Hence M is a solvable T^*-group by induction on $|G|$. Since M is an arbitrary maximal subgroup of G, all maximal subgroups of G are solvable T^*-groups. By

Theorem 2.3.14, all proper subgroups of G are solvable T^*-groups. If G is not a T^*-group, then there exist a normal Sylow p-subgroup P of G and a cyclic Sylow p-subgroup of G, where p and q are distinct primes, such that $G = PQ$ by Corollary 2.3.18. Clearly, $P = H$ and so $G \,/\, H \cong Q$. Since Q is cyclic, it follows easily that $G \,/\, H$ is a solvable T-group. Now [As4, p. 42] implies that G is a solvable T-group, so G is a solvable T^*-group, a contradiction. Thus G is a solvable T^*-group. □

Theorem 2.3.20 [ACs4, Theorem 7]**.** *G is a solvable T^*-group if and only if $G = M \cdot N$, where N is a nilpotent Hall subgroup of G and M is a nilpotent normal Hall subgroup of G such that all subgroups of prime power order of M are pronormal in G.*

Proof. Let G be a solvable T^*-group. By Theorem 2.3.15, $G = HK$, where H is a nilpotent normal Hall subgroup and K is a nilpotent Hall subgroup of G. Let Q be a Sylow q-subgroup of H. We show that one of the following is true:

a) Every subgroup of Q is normal in G.

b) Every Sylow r-subgroup ($r \neq q$) of G is centralizing Q.

Denote $H = Q \times T$. Consider $L_0 = T \cdot K$. Let L be the normal closure of L_0. We show that every subgroup of Q is normal in L. Let Q_0 be an arbitrary subgroup of Q. As Q_0 is subnormal in G, Q_0 is S-quasinormal in G, consequently $Q_0 \cdot R = R \cdot Q_0$ for all Sylow subgroups R of G. Since Q is normal in G, $R \leqq N_G(Q_0)$ follows by Lemma 2.3.11. Thus $L \leqq N_G(Q_0)$.

If $L = G$, then $N_G(Q_0) = G$, so $L \lneqq G$. As $G = L_0 \cdot Q$ and $L_0 < L$, $Q \nleqq L$ follows. Denote $Q_1 = Q \cap L$. We have $L \lhd G$, $Q \lhd G$. Clearly each element of L_0 induces the identity on Q/Q_1 by conjugation. We saw that L_0 normalizes every subgroup of Q. Applying Lemma 2.3.12 $L_0 \leqq C_G(Q)$ follows. So $G = Q \times L_0$. Thus every Sylow r-subgroup ($r \neq q$) centralizes Q.

Let H_1 be the direct product of those Sylow subgroups of H whose subgroups are all normal in G, and let H_2 be the direct product of those Sylow subgroups of H which are centralized by all of the other Sylow subgroups. Denote $M = H_1$ and $N = H_2 \cdot K$. Clearly $G = M \cdot N$, M is a nilpotent normal Hall subgroup and all subgroups of prime power order of M are normal in G, consequently they are pronormal in G. Obviously N is a nilpotent Hall subgroup of G.

Vice versa, if $G = M \cdot N$ with the above properties, then $G/M \cong N$ and the nilpotency of N yields that G/M is a solvable T^*-group. Furthermore all subgroups of prime power order of M are pronormal in G. Applying Theorem 2.3.19 it follows that G is a solvable T^*-group. □

Theorem 2.3.21 [ACs4, Theorem 8]**.** *If a group G possesses three solvable T^*-groups, whose indices are pairwise relatively prime, then G is a solvable T^*-group.*

Proof. Let H_i, $1 \leqq i \leqq 3$, be the three given subgroups of G. If $H_1 = 1$, then $|G : H_1| = |G|$. Then $|G : H_2|$ must be relatively prime to $|G|$, which is possible only if $H_2 = G$, whence G is a solvable T^*-group in this case. Hence we may assume that $H_i \neq 1$, $1 \leqq i \leqq 3$. Let M_i, $1 \leqq i \leqq 3$, be three maximal subgroups of G such that $H_i \leqq M_i$. We argue by induction on $|G|$ that each M_i is a solvable T^*-group. By [Go, Theorem 4.4, p. 232], G is solvable. Now by [Go, Theorem 1.5, p. 219], $|G : M_1| = p_1^{e_1}$, where p_1 is a prime. Since M_1, H_2 and H_3 have pairwise relatively prime indices, it follows that $G = M_1H_2 = M_1H_3$ and so $|G : H_2| = |M_1 : M_1 \cap H_2|$ and $|G : H_3| = |M_1 : M_1 \cap H_3|$. By Theorem 2.3.14, $M_1 \cap H_2$ and $M_1 \cap H_3$ are solvable T^*-groups. Hence by induction, M_1 is a solvable T^*-group. Similarly, M_2, M_3 are solvable T^*-groups. Let N be a maximal subgroup of G. If $N = M_1^x$ for some x in G, then N is a solvable T^*-group. Now we can assume that $N \neq M_i^x$, $1 \leqq i \leqq 3$, for any x in G. Then [Hu1, Theorem 3.9, p. 165] implies that $G = NM_i$, $1 \leqq i \leqq 3$, and so $|G : M_i| = |N : N \cap M_i|$. Obviously, $|G : M_i| = p_i^{e_i}$, where p_1, p_2, p_3 are distinct primes. Also $N \cap M_1$, $N \cap M_2$ and $N \cap M_3$ are solvable T^*-groups by Theorem 2.3.14. Hence by induction, N is a solvable T^*-group. Since N is an arbitrary maximal subgroup of G, it follows that all maximal subgroups of G are solvable T^*-groups. Now by Theorem 2.3.14, all proper subgroups of G are solvable T^*-groups. If G is not a T^*-group, then $|\pi(G)| = 2$ by Theorem 2.3.16, and this is impossible as the indices of H_i, $1 \leqq i \leqq 3$, in G are pairwise relatively prime. Thus G is a solvable T^*-group. □

Theorem 2.3.22 [ACs4, Theorem 9]**.** *G is a solvable T^*-group if and only if G is a T^*-group and all of whose proper subgroups are T^*-groups.*

Proof. If G is a solvable T^*-group, the conclusion follows from Theorem 2.3.14. To prove the converse of the theorem, we need only show that G is solvable. Let p be the smallest dividing the $|G|$ and let P be a Sylow p-subgroup of G. If there is a proper subgroup H of P such that $H \triangleleft G$, then G / H is solvable by induction on $|G|$ and so G is solvable. Hence $N_G(H) < G$ for any proper subgroup H of P and so $N_G(H)$ is solvable by induction on $|G|$. Since $N_G(H)$ is a solvable T^*-group, it follows that $N_G(H)$ is supersolvable and so $N_G(H)$ has a normal p-complement. Then by Frobenius' Theorem [Go], G has a normal p-complement. It follows now from the Feit–Thompson Theorem [FT] that G is solvable. □

Continuing the study of solvable T^*-groups I generalized Zacher's theorem concerning solvable T-groups.

In paper [Cs3] I used world quasinormality for S-quasinormality, but here I use S-quasinormality.

Theorem 2.3.23 [Cs3, Theorem 1]**.** *Let G be a solvable finite group, the prime divisors of its order $p_1 > p_2 > \cdots > p_k$, and let $P_1, \ldots, P_k$ be a Sylow system with*

$P_i \in \mathrm{Syl}_{p_i}(G)$. *$G$ is a solvable T^*-group if and only if it satisfies the following conditions:*

(i) *If $1 \leqq i < j \leqq k$, then $P_j \leqq N_G(P_i)$.*

(ii) *For all $1 \leqq i < j \leqq k$, if $x \in P_i$, $y \in P_j$, then there exists a natural number n such that $x^y = x^n$.*

Proof. 1. Suppose G is a solvable T^*-group. Then by Lemma 2.3.10 G is supersolvable whence it has a Sylow tower. So G satisfies (i). As every subgroup of a solvable T^*-group is again a T^*-group by Theorem 2.3.14, it follows that $N_G(P_i)$ is a solvable T^*-group. We have that $\langle x \rangle$ is subnormal in $N_G(P_i)$, and using Lemma 2.3.11 $P_j < N_G(\langle x \rangle)$ is true. Thus G satisfies (ii).

2. Conversely, assume G satisfies (i) and (ii). First we show that G is supersolvable. Let L be a subgroup of order p_1 in $Z(P_1)$. Using (i), (ii) $L \trianglelefteq G$ follows. Then G/L is supersolvable by induction on $|G|$. We show that every subgroup of P_i is S-quasinormal in $N_G(P_i)$. Let B be an arbitrary subgroup of P_i. By the conditions $P_j < N_G(B)$ for all $j > i$. As $P_j^y < N_G(B^y)$ for every $y \in N_G(P_i)$, clearly any Sylow p_j-subgroup of $N_G(P_i)$ normalizes any subgroup of P_i. Let $\ell < i$ and let D be an arbitrary Sylow p_ℓ-subgroup of $N_G(P_i)$. By Hall's theorems $(P_iD)^z \leqq P_\ell P_i$ for some $z \in G$. As $(P_iD)^z = P_i^z D^z$, $D^z \leqq P_\ell$ and $P_i < N_G(P_\ell)$ clearly $P_i^z < N_G(D^z)$ follows, furthermore $D^z \leqq N_G(P_i^z)$, whence D centralizes P_i.

Thus every subgroup of P_i is S-quasinormal in $N_G(P_i)$. Using Lemma 2.3.13 it follows that either $P_i \leqq G'$ or each Sylow q-subgroup ($q \neq p_i$) of $N_G(P_i)$ centralizes P_i. By the supersolvability of G, G' is nilpotent, and we can repeat a part of proof of Theorem 2.3.15. Let H be the direct product of every P_j, $1 \leqq j \leqq k$ such that $P_j \leqq G'$. So H is a nilpotent normal Hall subgroup of G. Let P_ℓ, P_m, $1 \leqq \ell, m \leqq k$, $\ell \neq m$ be such that $P_\ell, P_m \not\leqq G'$. Assume $\ell < m$, then P_m is a Sylow subgroup of $N_G(P_\ell)$. By the above P_m centralizes P_ℓ.

Let K be the direct product of every P_h with $1 \leqq h \leqq k$ and $P_h \not\leqq G'$. Clearly K is a nilpotent Hall subgroup of G, furthermore $G = HK$, $H \cap K = 1$. The nilpotency of H and K and (ii) imply that for arbitrary $x \in H$ and $y \in K$ there exists a natural number n such that $x^y = x^n$. Then G is a solvable T^*-group by Theorem 2.3.15. $\square$

We need the following

Lemma 2.3.24 [Cs3, Lemma]**.** *Let U be a p-subgroup of G, $a \in N_G(U)$ such that $(o(a), |U|) = 1$ furthermore a normalizes every subgroup of U. If there is an element $b \neq 1$ of U such that $ab = ba$, then $a \in C_G(U)$ follows.*

Proof. Clearly $C_U(a) \neq 1$. Assume $C_U(a) \neq U$.

(a) $Z(\Omega_1(U)) \not\geqq C_U(a)$.

Denote $W = Z(\Omega_1(U))C_U(a)$. Clearly $\langle a \rangle$ normalizes every subgroup of W and each element of $\langle a \rangle$ induces the identity on $W/Z(\Omega_1(U))$ by conjugation. Applying

Lemma 2.3.12 $\langle a\rangle \leqq C_G(W)$ follows, a contradiction.

(b) $Z(\Omega_1(U)) > C_U(a)$.

Clearly there exists a subgroup $T \neq 1$ such that $Z(\Omega_1(U)) = C_U(a) \times T$. Let $b \in T$, $b \neq 1$ and $u \in C_U(a)$, $u \neq 1$. Clearly a normalizes $\langle b\rangle$ and $\langle bu\rangle$, consequently $(bu)^a = (bu)^m$ where $2 \leq m \leqq p-1$. As $(bu)^a = b^a u = b^m u^m$, $u^{m-1} = (b^m)^{-1} b^a$ follows. We have $b^a = b^n$ where $2 \leqq n \leqq p-1$ so $u^{m-1} = b^{n-m}$ is true, but $\langle u\rangle \cap \langle b\rangle = 1$ thus $u^{m-1} = 1$, a contradiction.

So $Z(\Omega_1(U)) = C_U(a)$.

(c) $Z(\Omega_1(U)) < \Omega_1(U)$.

Similarly to case (b) we can show that this case is impossible too.

Thus $C_U(a) = Z(\Omega_1(U)) = \Omega_1(U)$. Let $\ell \in U \setminus \Omega_1(U)$. By the conditions a normalizes $\langle \ell\rangle$. As a centralizes $\Omega_1(\langle \ell\rangle)$ consequently a centralizes $\langle \ell\rangle$. A contradiction. □

We give a characterization of solvable T^*-groups by the normalizers of certain p-subgroups.

Theorem 2.3.25 [Cs3, Theorem 2]**.** *G is a solvable T^*-group if and only if every p-subgroup A (for all prime divisors p of the order of G) is S-quasinormal in $N_G(P_0)$ where P_0 is a p-subgroup containing the subgroup A.*

Proof. Assume G is a solvable T^*-group. Then by Theorem 2.3.14 $N_G(P_0)$ is a solvable T^*-group too. Clearly A is subnormal in $N_G(P_0)$, whence A is S-quasinormal in $N_G(P_0)$.

Conversely, let p_1 be the smallest prime divisor of the order of G. We show that G has a normal p_1-complement. Let P_1 be a Sylow p_1-subgroup of G and let H be an arbitrary subgroup of P_1. We prove that $N_G(H)/C_G(H)$ is a p_1-group. Assume there is an element b of $N_G(H) \setminus C_G(H)$ of order q with $q \neq p_1$. Let a be an element of H of order p_1. By the conditions $\langle a\rangle$ is S-quasinormal in $N_G(H)$. It is easy to see $b \in N_H(\langle a\rangle)$. As $q > p_1$, $b \in C_G(a)$ follows. Clearly every subgroup of H is S-quasinormal in $N_G(H)$ by the conditions, whence b normalizes every subgroup of H. Using our Lemma 2.3.24 $b \in C_G(H)$ is true, a contradiction. Thus $N_G(H)/C_G(H)$ is a p_1-group, consequently G has a normal p_1-complement. So $G = P_1 K$, $K \triangleleft G$ and $P_1 \cap K = 1$. Consider the smallest prime divisor p_2 of the order of K. Similarly we can prove that K has a normal p_2-complement.

Thus G has a tower such that the prime divisors of the order of G are $p_1 < p_2 < \cdots < p_k$ and for arbitrary $1 \leqq i \leqq k$ there is a Sylow p_i-subgroup such that $P_i < N_G(P_j)$ for all $1 \leqq i < j \leqq k$. If i and j are such as above and $x \in P_j$, $y \in P_i$, then $\langle x\rangle$ is S-quasinormal in $N_G(P_j)$ by the conditions, whence it is easy to see that $y \in N_G(\langle x\rangle)$, consequently $x^y = x^n$ for some natural number n. Applying Theorem 2.3.23 G is a solvable T^*-group. □

6 years later Ballester-Bolinches and Esteban-Romero in [BR] introduced the following notion:

Definition. Let p be a prime number. A group G is said to be a Y_p-group when for all p-subgroups H and K of G such that $H \leqq K$, H is S-permutable (S-quasinormal) in $N_G(K)$.

After studying the properties of Y_p-groups they characterized the solvable PST-groups (T^*-groups) and they received the same result as Theorem 2.3.25 which is a six years earlier result in another form:

Theorem 2.3.28 [BR, Theorem 4]**.** *A group G is a solvable PST-group if and only if G satisfies Y_p for all primes p.*

We characterize the subnormal p-subgroups of a solvable T^*-group.

Theorem 2.3.26 [Cs3, Theorem 3]**.** *Let G be a solvable T^*-group. Then an arbitrary subnormal p-subgroup of G (for all prime divisors p of the order of G) is either normal or it is centralized by all Sylow q-subgroups of G with $q \neq p$.*

Proof. Let A be a subnormal p-subgroup of G. By Theorem 2.3.20 $G = MN$ where M is a nilpotent normal Hall subgroup of G, N is a nilpotent Hall subgroup of G, $M \cap N = 1$, furthermore every subgroup of prime power order of M is normal in G.

(a) $A \leqq M$. By the above A is normal in G.

(b) $A \leqq N^y$ for some $y \in G$.

Let Q be a Sylow q-subgroup of M with $q \neq p$. By the subnormality of A there is a chain $A \lhd A_1 \lhd \ldots \lhd A_\ell \lhd A_{\ell+1} \lhd \ldots \lhd A_m = G$.

Let A_ℓ be such that $Q \leqq A_\ell$ but $Q \nleqq A_{\ell-1}$. A normalizes every subgroup of Q. Since $A_{\ell-1} \lhd A_\ell$, $Q \lhd A_\ell$ and $A \leqq A_{\ell-1}$ it follows that each element of A induces the identity on $Q/Q \cap A_{\ell-1}$ by conjugation. Using Lemma 2.3.12 $A \leqq C_G(Q)$ follows. As Q is an arbitrary Sylow subgroup of M, $A \leqq C_G(M)$ is true. We have $G = M \cdot N^y$, N^y is a nilpotent Hall subgroup of G and $N^y = P \times T$ where P is a Sylow p-subgroup of G, whence $C_G(M) \geqq M \cdot T$. As $MT \lhd G$ it is easy to see that A is centralized by an arbitrary Sylow q-subgroup of G with $q \neq p$. □

Finally we get a newer characterization theorem for solvable T^*-groups by the description of the properties of Sylow subgroups:

Theorem 2.3.27 [Cs3, Theorem 4]**.** *G is a solvable T^*-group if and only if every Sylow subgroup P satisfies one of the following conditions:*

(a) *every subgroup of P is normal in G;*

(b) *every Sylow subgroup of $N_G(P)$ different from P centralizes P.*

Proof. Assume G is a solvable T^*-group. By Theorem 2.3.20 $G = MN$ where M is a nilpotent normal Hall subgroup of G, N is a nilpotent Hall subgroup of G, $M \cap N = 1$ and every subgroup of prime power order of M is pronormal in G. Let R be an arbitrary Sylow subgroup of G.

Assume $R \leqq M$. Clearly every subgroup of R is subnormal in G, on the other hand by the above they are pronormal in G, consequently every subgroup of R is normal in G.

Assume $R \leqq N^y$ for some $y \in G$. Clearly $N_G(R) = N^y \cdot (N_G(R) \cap M)$. The structure of G yields $B = N_G(R) \cap M \leqq C_G(R)$ and $N^y = R \times L$ where L is a nilpotent Hall subgroup of G, consequently $N_G(R) = R \times (L \cdot B)$ so R satisfies (b).

Conversely, let G be a counterexample of smallest order. Let M_0 be the product of every Sylow subgroup of G each subgroup of which is normal in G. Clearly M_0 is a nilpotent normal Hall subgroup of G. By the Theorem of Zassenhaus there is a subgroup N_0 such that $M_0 \cdot N_0 = G$ and $M_0 \cap N_0 = 1$. Clearly N_0 is a Hall subgroup in G and it satisfies the conditions of our theorem. By the minimality of G N_0 is a solvable T^*-group. Using Theorem 2.3.15, $N_0 = A \cdot B$ where A is a nilpotent normal Hall subgroup of G, B is a nilpotent Hall subgroup of G and $A \cap B = 1$, furthermore for arbitrary $a \in A$, $b \in B$ there is a natural number i such that $a^b = a^i$. Assume $A \neq 1$. Let P be a Sylow subgroup of A. Using $N_{N_0}(P) = N_0$, $B \leqq C_G(P)$ follows by the conditions whence $B \leqq C_G(A)$. Thus $N_0 = A \times B$ and N_0 is a nilpotent normal Hall subgroup of G. Using Theorem 2.3.15 it follows that G is a solvable T^*-group. $\square$

By studying new properties of PT- and PST-groups, this topic is the subject of extensive research. See [ABRP], [ABP], [Ra2], [BRR], [BR], [Ro].

References

[Ag] Agrawal, R. K.: Finite groups whose subgroups permute with all Sylow subgroups, *Proc. Amer. Math. Soc.* 47 (1975), 77–83.

[A1] Albert, A. A.: Quasigroups I, *Trans. Amer. Math. Soc.* 54 (1943), 507–519.

[A2] Albert, A. A.: Quasigroups II, *Trans. Amer. Math. Soc.* 55 (1944), 401–419.

[ABP] Alejandre, M. J.; Ballester-Bolinches, A.; Pedraza-Aguilera, M. C.: Finite Soluble Groups, *J. Algebra* 240 (2001), 705–722.

[ABRP] Alejandre, M. J.; Ballester-Bolinches, A.; Esteban-Romero, R.; Pedraza-Aguilera, M. C.: Permutability and subnormality in finite groups, *Groups St. Andrews 2001 in Oxford*, London Math. Soc. Lecture Note Series 304, 1–11.

[MS] Al-Mosa Al-Shomrani, M. M.: On minimal non PT-groups, *Intern. Math. Forum* 3 no. 10 (2008), 495–501.

[As1] Asaad, M.: On p-Nilpotence and Supersolvability of Finite Groups, *Comm. Algebra* 34 (2006), 189–195.

[As2] Asaad, M.: Some results on p-nilpotence and supersolvability of finite groups, *Comm. Algebra* 34 (2006), 4217–4224.

[As3] Asaad, M.: Finite groups in which normality or quasinormality is transitive, *Archiv* 83 (2004), 289–296.

[As4] Asaad, M.: Investigations in the theory of finite groups, Dissertation for Candidate Degree, Budapest, 1976.

[AsBP] Asaad, M.; Ballester-Bolinches, A.; Pedraza-Aguilera, M. C.: A note on minimal subgroups of finite groups, *Comm. Algebra* 24 (8) (1996), 2771–2776.

[ACs1] Asaad, M.; Csörgő, P.: The influence of minimal subgroups on the structure of finite groups, *Arch. Math.* 72 (1999), 401–404.

[ACs2] Asaad, M.; Csörgő, P.: Characterization of Finite Groups with Some S-quasinormal Subgroups, *Monatsh. Math.* 146 (2005), 263–266.

[ACs3] Asaad, M.; Csörgő, P.: Some results on supersolvability of finite groups, *Monatsh. Math.* 154 (2008), 265–269.

[ACs4] Asaad, M.; Csörgő, P.: On T^*-groups, *Acta Math. Hungar.* 74 (3) (1997), 235–243.

[AsH] Asaad, M.; Heliel, A. A.: On permutable subgroups of finite groups, *Arch. Math.* 80 (2003), 113–118.

[AsR] Asaad, M.; Ramadan, M.: On the intersection of maximal subgroups of a finite group, *Archiv Math.* 61 (1993), 206–214.

[ARS] Asaad, M.; Ramadan, M.; Shaalan, A.: Influence of π-quasinormality of maximal subgroups of Sylow subgroups of Fitting subgroups of a finite group, *Arch. Math.* 56 (1991), 521–527.

[Bae1] Baer, R.: Situation der Untergruppen und Struktur der Gruppen, *Sitz.-Ber. Heidelberg, Akad. Wiss.* 2 (1933), 12–17.

[Bae2] Baer, R.: Erweiterung von Gruppen und ihren Isomorphismen, *Math. Z.* 38 (1934), 375–416.

[Bae3] Baer, R.: Nets and groups, *Trans. Amer. Math. Soc.* 46 (1939), 110–141.

[Bae4] Baer, R.: Groups with preassigned central and central quotient group, *Trans. Amer. Math. Soc.* 44 (1938), 387–412.

[BR] Ballester-Bolinches, A.; Esteban-Romero, R.: Sylow Permutable Subnormal Subgroups of Finite Groups, *J. Algebra* 251 (2002), 727–738.

[BRP] Ballester-Bolinches, A.; Esteban-Romero, R.; Pedraza-Aguilera, M. C.: On finite groups in which subnormal subgroups satisfy certain permutability conditions, *Adv. in Algebra*, Proceedings of ICM Satellite, 2003.

[BRR] Ballester-Bolinches, A.; Esteban-Romero, R.; Ragland, M.: A note on finite PST-groups, *J. Group Theory* 10 (2) (2007), 205–210.

[Bas] Basarab, A. S.: Klass LK-lup, *Matematicheskie issledovanija* 120 (1991), 3–7.

[BT] Best, E.; Taussky, O.: A class of groups, *Proc. Roy. Irish Acad. Sect. A.* 47 (1942), 55–62.

[BMHV] Bianchi, M.; Mauri, A. G. B.; Herzog, M.; Verardi, L.: On finite solvable groups in which normality is a transitive relation, *J. Group Theory* 3 (2000), 147–156.

[Br] Bruck, R. H.: Contributions to the theory of loops, *Trans. Amer. Math. Soc.* 60 (1946), 245–354.

[Buch] Buchsteiner, H. H.: O nekotorom klasse binarnych lup, *Mat. Issled.* 39 (1976), 54–66.

[Bu] Buckley, J.: Finite groups whose minimal subgroups are normal, *Math. Z.* 116 (1970), 15–17.

[Ca] Carr, J. G.: A solubility criterion for factorised groups, *Arch. Math.* 27 (1976), 225–231.

[Co] Collins, M. J.: *Representations and Characters of Finite Groups*, Cambridge University Press, 1990.

[Cs1] Csörgő, P.: On supersolvability of finite groups, *Glasgow Math. J.* 43 (2001), 327–333.

[Cs2] Csörgő, P.: On the natural factorization of finite supersolvable groups, Groups–Korea '98, *Proc. of Intern. Conf. at Pusan Nat. Univ.*, 91–94, Walter de Gruyter, Berlin–New York, 2000.

[Cs3] Csörgő, P.: The properties of T^*-groups, *Publ. Math.* 49/1-2 (1996), 93–97.

[Cs4] Csörgő, P.: On connected transversals to abelian subgroups and loop theoretical consequences, *Archiv Math.* 86 (2006), 499–516.

[Cs5] Csörgő, P.: Abelian inner mappings and nilpotency class greater than two, *Europ. J. Comb.* 28 (2007), 858–867.

[Cs6] Csörgő, P.: Extending the structural homomorphism of LCC loops, *Comment Math. Univ. Carolin.* 46, 3 (2005), 385–389.

[Cs7] Csörgő, P.: On loops that are abelian groups over the nucleus and Buchsteiner loops, *Comment. Math. Univ. Carolin.* 49, 2 (2008), 197–208.

[Cs8] Csörgő, P.: On π-T^*-groups, *J. Egyptian Math. Soc.* 8 (2) (2000), 121–126.

[Cs9] Csörgő, P.: On π-t-groups, *Publ. Math.* 35 (1988), 255–259.

[CsD1] Csörgő, P.; Drápal, A.: Left conjugacy closed loops of nilpotency class two, *Result Math.* 47 (2005), 242–265.

[CsD2] Csörgő, P.; Drápal, A.: On left conjugacy closed loops in which the left multiplication group is normal, *Abh. Math. Semin. Univ. Hamburg* 76 (2006), 17–34.

[CsD3] Csörgő, P.; Drápal, A.: On loops rich in automorphisms that are abelian module the nucleus, to appear in *Forum Mathematicum*.

[CsD4] Csörgő, P.; Drápal, A.: Buchsteiner loops and conjugacy closedness, to appear in *Comm. Algebra*.

[CsDK] Csörgő, P.; Drápal, A.; Kinyon, M.: Buchsteiner loops, submitted.

[CsH] Csörgő, P.; Herzog, M.: On Supersolvable Groups and the Nilpotator, Comm. Algebra 32 (2) (2004), 609–620.

[CsJK] Csörgő, P.; Jančařík, A.; Kepka, T.: Generalized capable abelian groups, *Nonassoc. Algebras and its Applications*, A series of Lecture Notes in Pure and Applied Mathematics 246, 129–136.

[CsK] Csörgő, P.; Kepka, T.: On loops whose inner permutations commute, *Comment. Math. Univ. Carolin.* 45 (2004), 213–221.

[CsMN] Csörgő, P.; Myllylä, K.; Niemenmaa, M.: On Connected Transversals to Dihedral Subgroups of Order $2p^n$, *Algebra Colloquium* 7: 1 (2000), 105–112.

[CsN1] Csörgő, P.; Niemenmaa, M.: Solvability Conditions for Loops and Groups, *J. Algebra* 232 (2000), 336–342.

[CsN2] Csörgő, P.; Niemenmaa, M.: On Connected Transversals to Nonabelian Subgroups, *Europ. J. Combinatorics* 23 (2002), 179–185.

[De] Dedekind, R.: Über Gruppen, deren sämtliche Teiler Normalteiler sind, *Math. Ann.* 48 (1897), 548–561.

[Do] Doerk, K.: Minimal nicht überauflösbare endliche Gruppen, *Math. Z.* 91 (1966), 198–205.

[DH] Doerk, K.; Hawkes, T.: *Finite soluble groups*, De Gruyter, Berlin–New York, 1992.

[D1] Drápal, A.: Conjugacy closed loops and their multiplication groups, *J. Algebra* 272 (2004), 838–850.

[D2] Drápal, A.: Orbits of Inner Mapping Groups, *Monatsh. Math.* 134 (2002), 191–206.

[D3] Drápal, A.: On multiplication groups of left conjugacy closed loops, *Comment. Math. Univ. Carolin.* 45 (2004), 223–236.

[DK] Drápal, A.; Kinyon, M.: Buchsteiner loops: associators and constructions, submitted.

[DV] Drápal, A.; Vojtěchovský, P.: Explicit constructions of loops with commuting inner mappings, *Europ. J. Combinatorics* 29/7 Oct. (2008), 1662–1681.

[FT] Feit, W.; Thompson, G. J.: Solvability of groups of odd order, *Pac. J. Math.* 13 (1963), 775-1029.

[Ga] Gaschütz, W.: Gruppen, in denen das Normalteilersein transitiv ist, *J. Reine Angew. Math.* 198 (1957), 87–92.

[Gla] Glauberman, G. G.: Fixed points in group with operators, *Math. Z.* 84 (1964), 120–125.

[GR] Goodaire, E. G.; Robinson, D. A.: A class of loops which are isomorphic to all loop isotopes, *Canad. J. Math.* 34 (1982), 662–672.

[Go] Gorenstein, D.: *Finite Groups*, New York, 1968.

[Ha] Hall, M.: *The theory of groups*, Macmillan, New York, 1959.

[HLL] Heliel, A. A.; Li, Xianhua; Li, Yangming: On 3-permutability of minimal subgroups of finite groups, *Arch. Math.* 83 (2004), 9–16.

[Hu1] Huppert, B.: *Endliche Gruppen I*, Berlin–Heidelberg–New York, 1967.

[Hu2] Huppert, B.: *Endliche Gruppen I*, Berlin–Heidelberg–New York, 1979.

[HuB] Huppert, B.; Blackburn, N.: *Finite groups III*, Berlin–Heidelberg–New York, Springer, 1982.

[Ke] Kegel, O. H.: Sylow-Gruppen und Subnormalteiler endlicher Gruppen, *Math. Z.* 78 (1962), 205–221.

[K1] Kepka, T.: On the abelian inner permutation groups of loops, *Comm. Algebra* 26 (3) (1998), 857–861.

[KN1] Kepka, T.; Niemenmaa, M.: On loops with cuclic inner mapping groups, *Arch. Math.* 60 (1993), 233–236.

[KPh] Kepka, T.; Phillips, J. D.: Connected transversals to subnormal subgroups, *Comment. Math. Univ. Carolin.* 38, 2 (1997), 223–230.

[KiN] Kiechle, H.; Nagy, G. P.: On the extension of involutorial Bol loops, *Abh. Math. Semin. Univ. Hamburg* 72 (2002), 235–250.

[KKP] Kinyon, M. K.; Kunen, K.; Phillips, J. D.: Diassociativity in conjugacy closed loops, *Comm. Algebra* 32 (2004), 767–786.

[Ku] Kunen, K.: The structure of conjugacy closed loops, *Trans. Amer. Math. Soc.* 352 (2000), 2889–2911.

[La] Laue, R.: Dualization for saturation for locally defined formations, *J. Algebra* 52 (1978), 347–353.

[Ya] Li, Yangming: Finite group with some NE-subgroup, *Journal of Group Theory* 9, No. 1 (2006), 49–58.

[LW1] Li, Y.; Wang, Y.: The influence of minimal subgroups on the structure of a finite group, *Proc. Amer. Math. Soc.* 131 (2) (2003), 337–341.

[LW2] Li, Y.; Wang, Y.: On π-quasinormally embedded subgroups of finite groups, *Journal of Algebra* 281 (1) (2004), 109–123.

[M1] Moufang, R.: Alternativkörper und der Satz von vollständigen Vierseit, *Abh. Math. Semin. Hamburg Univ.* 9 (1933), 207–222.

[M2] Moufang, R.: Zur Struktur von Alternativkörpern, *Math. Annalen* 110 (1934), 416–430.

[M] Myllylä, K.: On connected transversals to dihedral subgroups, *Acta Univ. Oul.* A-350, 2000.

[MN] Myllylä, K.; Niemenmaa, M.: On the solvability of commutative loops and their multiplication groups, *Comment. Math. Univ. Carolin.* 40, 2 (1999), 209–213.

[GN] Nagy, G. P.: Group invariants of certain Burn loop classes, *Bull. Belg. Math. Soc.* 5 (1998), 403–415.

[NStr] Nagy, P.; Strambach, K.: Loops as invariant sections in groups and their geometry, *Canad. J. Math.* 46 (1994), 1027–1056.

[GNV] Nagy, G. P.; Valsecchi, M.: On nilpotent Moufang loops with central associators, *J. Algebra* 307 (2007), 547–567.

[NV1] Nagy, G. P.; Vojtěchovský, P.: Computing with small quasigroups and loops, *Quasigroups and Related Systems* 15 (2007), 77–94.

[NV2] Nagy, G. P.; Vojtěchovský, P.: Moufang loops with commuting inner mappings, submitted.

[NST] Neumann, P. M.; Stoy, G.; Thompson, E.: *Groups and Geometry*, Oxford Univ. Press, London, 1994.

[N1] Niemenmaa, M.: On finite loops and their inner mapping groups, *Comment. Math. Univ. Carolin.* 45, 2 (2004), 341–347.

[N2] Niemenmaa, M.: On the relation between group theory and loop theory, *Groups St. Andrews 2001 in Oxford*, London Math. Soc. Lecture Note Series 305, 422–427.

[N3] Niemenmaa, M.: On the structure of the inner mapping groups of loops, *Comm. Algebra* 24 (1996), 135–142.

[N4] Niemenmaa, M.: On finite loops whose inner mapping groups are abelian, *Bull. Austr. Math. Soc.* 65 (2002), 477–484.

[N5] Niemenmaa, M.: Transversals, Commutators and Solvability in Finite Groups, *Bolletino U. M. I.* (7) 9-A (1995), 203–208.

[N6] Niemenmaa, M.: On Connected Transversals to Subgroups Whose Order is a Product of Two Primes, *Europ. J. Combinatorics* 18 (1997), 915–919.

[N7] Niemenmaa, M.: On loops which have dihedral 2-groups as inner mapping groups, *Bull. Austral. Math. Soc.* 52 (1995), 153–160.

[N8] Niemenmaa, M.: On the structure of finite loop capable Abelian groups, *Comment. Math. Univ. Carolin.* 48, 2 (2007), 217–228.

[NK1] Niemenmaa, M.; Kepka, T.: On Multiplication Groups of Loops, *J. Algebra* 135 (1) (1990), 112–122.

[NK2] Niemenmaa, M.; Kepka, T.: On connected transversals to abelian subgroups, *Bull. Austr. Math. Soc.* 49 (1994), 121–128.

[NK3] Niemenmaa, M.; Kepka, T.: On connected transversals to abelian subgroups in finite groups, *Bull. London Math. Soc.* 24 (1992), 343–346.

[NR] Niemenmaa, M.; Rytty, M.: Connected transversals and multiplication groups of loops, *Quasigroups and Related Systems* 15 (2007), 95–107.

[Or] Ore, O.: Contributions to the theory of groups of finite orders, *Duke Math. J.* 5 (1939), 431–460.

[Pe] Peng, T. A.: Finite groups with pro-normal subgroups, *Proc. Amer. Math. Soc.* 20 (1969), 232–234.

[Ra1] Ramadan, M.: The Influence of Certain Abelian Subgroups on the Structure of Finite Groups, *Alg. Coll.* 15, 3 (2008), 479–484.

[Ra2] Ramadan, M.: Finite groups in which permutability is a transitive relation on their Frattini factorgroups, *Acta Math. Acad. Sci. Hungar.* 114 (3) (2007), 187–193.

[Ro1] Robinson, D. J. S.: Finite groups in which normality or permutability is transitive, *Advances in Group Theory 2002*, Proc. Intensive Bimester Napoli (Italy), May–June 2002.

[Ro2] Robinson, D. J. S.: A note on finite groups in which normality is transitive, *Proc. Amer. Math. Soc.* 19 (1968), 933–937.

[Ro3] Robinson, D. J. S.: *A Course in the Theory of Groups*, Springer-Verlag, New York, 1982.

[Sha] Shaalan, A.: The influence of π-quasinormality of some subgroups on the structure of a finite group, *Acta Math. Hungar.* 56 (1990), 287–293.

[Shi] Shirong, L.: On minimal subgroups of finite groups, *Comm. Algebra* 22 (1994), 1913–1918.

[So] Soikis, L. R.: On specialnych lupach, in: *Voprosy teorii kvazigrupp i lup* (V. D. Belousov, ed.), Akademia Nauk Moldav. SSR, Kishinyev, 1970, pp. 122–131.

[Sr] Srinivasan, S.: Two sufficient conditions for supersolvability of finite groups, *Israel J. Math.* 35 (1980), 210–214.

[Yo1] Yokoyama, A.: Finite solvable groups whose F-hyper-center contains all minimal subgroups, *Arch. Math.* 26 (1975), 123–130.

[Yo2] Yokoyama, A.: Finite solvable groups whose F-hyper-center contains all minimal subgroups II, *Arch. Math.* 27 (1976), 572–575.

[Ve1] Vesanen, A.: Solvable Groups and Loops, *J. Algebra* 180 (1996), 862–876.

[Ve2] Vesanen, A.: The group $PSL(2, q)$ is not the multiplication group of a loop, *Comm. Algebra* 22 (4) (1994), 1177–1195.

[Wa] Wang, Y.: The Influence on Minimal Subgroups on the Structure of Finite Groups, *Acta Mathematica Sinica* 16 (1) (2000), 63–70.

[WW] Wang, Li Fang; Wang, Yan Ming: A Remark on 3-Permutability of Finite Groups, *Acta Math. Sinica, Engl. Series* 23 No. 11 (2007 Nov.), 1985–1990.

[We] Weinstein, M.: *Between nilpotent and solvable*, Passaic, 1982.

[Za] Zacher, G.: Caratterizzazione dei t-gruppi finite resolubili, *Richerche Mat.* 1 (1952), 287–294.